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MTM 21012 MATHEMATICAL MODELING
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1 INTRODUCTION

1.1 What Is Mathematical Modeling?

Real life problems arise from different disciplines- Engineering, sociology, chemistry, biology, physics, management, finance etc. From our school classes, we have been solving problems related to the real-world using some formulae.

Example on Mechanics

Consider the situation of total external force $\underline{F}$ acting on an object of constant mass m which is moving with the acceleration $\underline{a}$ is formulated by use of Newton's Second law as

$$\underline{F} = m\,\underline{a}.$$

This formula is an example for a **mathematical model** or simply a **model**.

Now let's recall the process of formulating this equation. During this process, we made some assumptions such as the mass is taken as constant and used some physical laws from literature that the net external force is directly proportional to the acceleration of the object. To find the proportionality constant, one can do experiments in the laboratory and can observe that it is nothing but mass. This process is known as **Mathematical Modeling**,

This model can be now used in solving problems of motion as far as the assumptions are valid. For example, consider the situation of falling an object of a mass m under gravitational field. We assume the acceleration $\underline{g}$ due to the gravitational field is constant and the resistance forces such as air resistance is ignored. Then the only force acting on the mass is its weight. Applying the formula, the weight is

$$\underline{w} = m\underline{g},$$

where $\underline{g} = 9.81\, ms^{-2}$.

However, when the assumptions are not met, we cannot use these models to understand the situation. For example, in the motion of rockets, the mass continuously changes and hence the model $\underline{F} = m\,\underline{a}$ is no longer useful. In this case, Newton's second law is the net external force is the rate of change of the momentum. That is,

$$\underline{F} = \frac{d}{dt}(m\underline{v}) = m\frac{d}{dt}\underline{v} + \underline{v}\frac{dm}{dt},$$

where the mass $m = m(t)$, $\underline{v} = \underline{v}(t)$ is the velocity of the object.

Note that when m is constant, $\frac{dm}{dt} = 0$ and hence $\underline{F} = m\frac{d}{dt}\underline{v} = m\,\underline{a}$.

That is, when a pre-found model does not illustrate the physical phenomena, we have to go back, revise the assumptions and theoretical knowledge and formulate a new model that suits to the new phenomena.

Example on Finance

Another simple model is finding the balance B of an account where the capital P is deposited in a bank which provides a simple interest at the annual rate r after n years. This can be formulated as

$$B = P + Prn = P(1 + rn).$$

The assumptions made here is no more deposits or withdrawals made during this year and the bank did bot change its interest rate. Also, the interest is given to the initial deposit only.

Takin this as basic model, we can now develop a model for complicated situations. Say, for example, Deposits and withdrawals are possible at anytime and interest is paid at the 1st day of each month depending on the balance of the last day of previous month (compound interest).

These models can be used to plan your financial needs and the model can be validated by checking the balance at your bank without waiting until you receive the amount you need.

Definition 1.1 A mathematical model is a mathematical formulation that describes some real-life situation whose theoretical and numerical analysis provides insight, answers, and guidance useful for the originating application. The process of constructing mathematical models is called mathematical modelling

In mathematical modelling, we take a real-world problem and write it as an equivalent mathematical problem. We then solve the mathematical problem, and interpret its solution in terms of the real-world problem. After this we see to what extent the solution is valid in the context of the real-world problem. So, the stages involved in mathematical modelling are formulation, solution, interpretation and validation

Mathematical modeling can only be learnt by doing it. There are no set of rules and the *right* way to model. It can only be reached by familiarity with relevant examples.

1.2 Why Mathematical Models?

Mathematical Model

- enables to design or control a system by a thorough understanding of the system modeled.
- allows the efficient (quick and accurate) computing capabilities.
- can be used to test the effect of the changes in system without do any changes in the system.
- do the predictions and take appropriate decisions.

- allows for experimentation (simulation) when it is impossible or too expensive in the real world.
- provides a mode of effective communication in / to a group.

1.3 Classification of Mathematical Models

Mathematical Models can be classified according to:

(a) **the subject or discipline**: Mathematical Physics, Theoretical Physics, Theoretical Chemistry, Mathematical Biology, Mathematical Economics, Econometrics, Mathematical Sociology, Mathematical Psychology, Mathematical Medicine, Mathematical modeling in transportation, Mathematical Modeling in urban and regional development, Mathematical Modeling in water resources and so on.

(b) **the purpose**: Mathematical Modeling for description, Mathematical Modeling for prediction, Mathematical Modeling for optimization, Mathematical Modeling for control and so on.

(c) **the nature**: *Linear* or *Non-linear* Models (according as basic equations describing the problem are linear or nonlinear), *Static* or *Dynamic* (according as the time variation in the system are not or are taken into account), *Deterministic* or *Stochastic* (according as the chance factor are not or are taken into account), *Discrete* or *Continuous* (according as the variables are discontinuous or continuous).

(d) **the mathematical techniques used to solve**: Mathematical Modeling through classical algebra, through linear algebra and matrices, ordinary and partial differential equations, through ordinary and partial difference equations, through integral equations, through integro-difference equations, through graphs, through programming and so on

1.4 Building a Model

The modeling process can be discussed in 7 steps.

Step 1: Problem Identification

What is the problem you want to study? Read (literature) and ask questions about the problem. What exactly you want to know? What are the factors (variables, parameters, constants) influencing the identified problem? The variables (symbols used to represent the factor) the model seeks to explain are dependent variables. The remaining are independent variables. Classify each variable as dependent, independent or neither.

Step 2: Simplify the situation

Usually, we cannot build a mathematical model considering all variables identified above. The task here is to simplify the problem by reducing the number of independent variables with the aid of some assumptions. This can be done in two ways: First, think which independent variables are minimal relevant? Which can be ignored? Second, determine relationships, if any, among the remaining variables. When the problem is so complicated in which we cannot find the relationship among all the variables initially, divide the problem into sub-models. That is, study one or more variables separately. Think of laws or theorems relating these variables. If necessary, collect some data and analyse it to get some initial insight into this situation.

Step 3: Build the Model and Solve the Problem

Connect all the sub-models together and describe in mathematical terms (may be equations, inequalities etc.) the relationship among the variables.

Think of all possible methods (available or develop new) to solve the equations of the model. The methods may be analytic, numerical or simulation. Interpret the solution and try to find the *best* solution to the problem.

If the model cannot be solved, return to step 2 and make additional simplification assumptions. If still difficult, then return to step 1 and redefine the problem.

Step 4: Verify and Revise

To test the validity of the model, sometimes we have to collect data which is a time consuming and expensive process. Thus, before collecting data and test the model, we have to think

(i) whether the model answer the problem identified in step 1?

(ii) is the model in a practical sense? That is, can we collect the data to operate the model?

(iii) does the model make commonsense?

If the answers for all these three questions are “yes”, then we can test the model using data obtained from several observations. If model agrees with the available observations or data, then accept the model. If model does not agree, return to step 2 and examine the assumptions and change if necessary. Continue this process till a satisfactory model is obtained.

If the answers for the at least one of the three questions is “no”, revise the method used to solve the model.

Step 5: Deduce conclusions

Deduce conclusions from your model and test these conclusions against earlier data and additional data that may be collected. Be noted that a model does not become a law just because it is verified repeatedly.

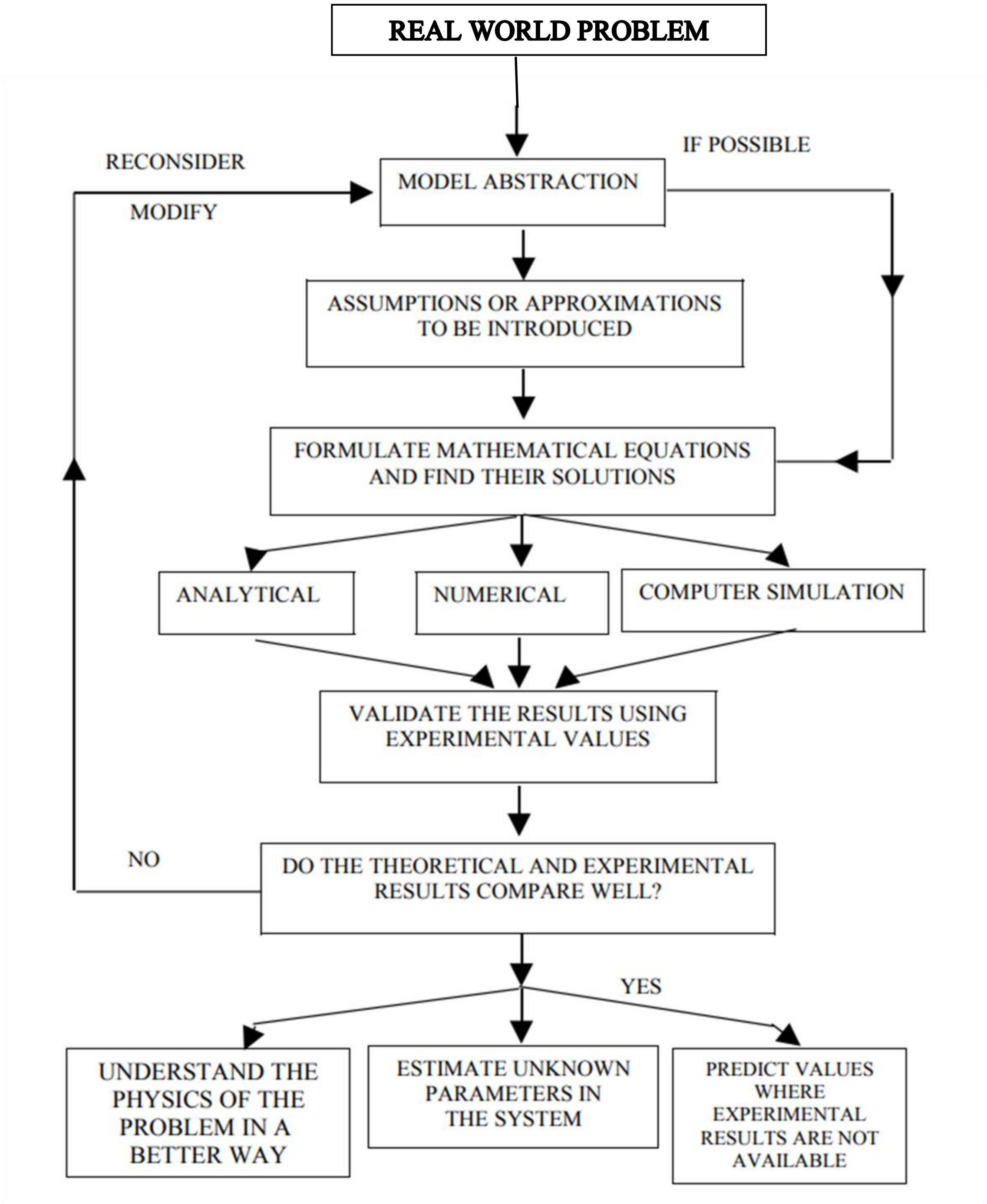
Step 6. Implement and Present

Explain and present the model so that the decision takers and users can understand if it is ever to be use to anyone. Make sure the model is in user friendly mode and less expensive.

Step 7: Maintain the Model

Be aware about the situation continuously. See always (i) Has original problem in step 1 changed in anyway? (ii) Have some previously neglected factors become important? (iii) Do some of the sub-models need to be adjusted?

FLOW CHART



Example 1. I travelled 432 kilometers on 48 liters of petrol in my car. Find a model to decide how much petrol do I need to travel a certain number of kilometers by my car, writing your assumptions, constants and variables clearly.
Use the model to find the amount of petrol I need to travel a place which is 520 km away and interpret your answer if I have only 12 liters petrol in my car.

Example 2. A carpenter produces and sells his own furniture. He produces a fixed number of each variety of furniture every year. Pine tables are sold for 650 $, cherry tables for 750 $ and maple tables for 850 $. He wants to find his total revenue in n years. List out the variables, constants and assumptions of the problem and obtain the model for him.

Use your model to find out the number of years he wants to wait to achieve his 1 million dollars revenue and interpret the results.

Example 3. The three phases of a project must be undertaken sequentially, which means that one phase cannot begin before the previous phase is finished. The cost of each of the phases breaks down into a fixed cost, independent of its duration, and a variable cost, which depends on the duration. The following table summarizes the situation:

PHASE	1	2	3
Fixed cost	318,000$	212,000 $	220,000 $
Variable Cost	15,000 $ / day	14,000 $ / day	16,000 $ / day

The designer of the project must propose a budget for the project. He would like to set a price that ensures a profit margin of at least 10%. Develop a model to express the total cost of the project and the price the designer should propose in function to the duration of each phase.

Example 4. In the year 2000, 191 member countries of the U.N. signed a declaration. In this declaration, the countries agreed to achieve certain development goals by the year 2015. These are called the *millennium development goals*. One of these goals is to promote gender equality. One indicator for deciding whether this goal has been achieved is the ratio of girls to boys in primary, secondary and tertiary education. India, as a signatory to the declaration, is committed to improve this ratio. The data for the percentage of girls who are enrolled in primary schools is given in the following table.

year	**Enrollment (in %)**
1991 – 92	41.9
1992 – 93	42.6
1993 – 94	42.7
1994 – 95	42.9
1995 – 96	43.1
1996 – 97	43.2
1997 – 98	43.5
1998 – 99	43.5
1999 – 2000	43.6
2000 – 01	43.7
2001 - 02	44.1

Using this data, develop two models to describe the rate at which the proportion of girls enrolled in primary schools grew. Use each model to estimate the year by which the enrolment of girls will reach 50%. The models can be tabled by calculating mean of the difference between percentages at the

(i) at the end of each year,

(ii) during these periods.

Graph the given enrollment percentage, both estimated percentages on the same plot.
Which model suits best for the situation? Justify your answer

ASSIGNMENT 1

1. A square bottom box without cover is made from a material that costs \$ 0.75 per square meters for the sides and \$ 0.95 per square meters for the bottom. Build a mathematical model to express the total cost of the material required to construct the box in function of its width and height.
   Due to price fluctuation, the costs of the materials are expected to increase by 10% every year. Develop your model to describe the current situation. Find another variable that can influence the total cost of the construction of box.

2. Suppose Nimal has invested ` Rs. 5,000.00 at 6% simple interest per year. With the return from the investment, he wants to buy a laptop that costs ` Rs. 132,000.00. Establish a model to decide how long he has to wait to get enough money to buy a laptop? List out the assumption, constants and variables you use for your model.
   He founds that he has to wait for a long time and hence he decided to deposit Rs. 500.00 at the 1^{st} day of every month to the same account. Re-build the model and interpret how many months he can advance the purchasing.
   Suggest him (without mathematical works) how can he achieve his goals even faster for the same amount of deposit.

3. We have given the timings of the gold medalists in the 400-metre race from the time the event was included in the Olympics, in the table below. Construct a mathematical model relating the years and timings.

Year	Timing in Seconds
1964	52.01
1968	52.03
1972	51.08
1976	49.28
1980	48.88
1984	48.83
1988	48.65
1992	48.83
1996	49.11
2000	49.41

   Timing for the same event in 2004 and 2008 Olympics were given as 49.41 and 48.82 seconds respectively. Validate your model using these two years.
   Use the model to estimate the timing in the 2012 Olympics.