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**MTM 21012 MATHEMATICAL MODELING**
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# 2. MATHEMATICAL MODELING THROUGH ODE OF FIRST ORDER

## 2.1 Introduction

Mathematical modeling in terms of differential equations arises when the situation modeled involves some *continuous variable(s)* varying with respect to some other continuous variable(s) and we have some hypothesis about the instantaneous *rate of change* of dependent variable(s) with respect to independent variable(s).

When we have one dependent variable depending on one independent variable, we get a model in terms of an ODE of the first order.

## 2.2 Linear Growth Models in Biology

### 2.2.1 Population Growth Models

**Problem Identification:** Suppose we know the size of the population $P_0$ at some given time $t_0 = 0$. We are interested with predicting the population $P$ at some future time $t_1$.

**Problem Simplification:** Obviously, $P$ is a discrete variable. However, it makes the modeling easier if we assume that it is a continuous variable $P(t)$. This will not be a serious error when population size is large and we consider the change of the population $\Delta P$ in a tiny time interval $\Delta t$.

**Factors:** Let

$t$ be the time (independent variable)
$P(t)$ be the size of the population at time $t,\ 0 \leq t \leq t_1$ (dependent variable),
$b$ be the birth rate (number of births per individual per unit time (constant),
$d$ be the death rate (number of deaths per individual per unit time (constant),
$P_0$ be the initial population.

**Formulation:**

Number of births $= bP(t)\Delta t + O_1(t)$, where $O_1(t)$ is the birth error term and $O_1(t) \longrightarrow 0$ as $\Delta t \longrightarrow 0$.
Number of deaths $= dP(t)\Delta t + O_2(t)$, where $O_2(t)$ is the birth error term and $O_2(t) \longrightarrow 0$ as $\Delta t \longrightarrow 0$.

Population at future time $\Delta t =$ Population at time $t +$ births during time $\Delta t$
$- $ deaths during time $\Delta t$

That is,

$$P(t+\Delta t) = P(t) + bP(t)\Delta t + O_1(t) - dP(t)\Delta t - O_2(t).$$

$$\Longrightarrow \quad \frac{P(t+\Delta t) - P(t)}{\Delta t} = \frac{\Delta P}{\Delta t} = (b-d)\,P(t) + \frac{O(t)}{\Delta t},$$

where $O(t) = O_1(t) - O_2(t) \longrightarrow 0$ as $\Delta t \longrightarrow 0$.

Assuming $b$ and $d$ are constants and denoting $(b-d) = k,$ $k$ is called the growth constant.

We have average rate of change of the population over the time interval $\Delta t$ is

$$\frac{\Delta P}{\Delta t} = k\,P(t)$$

and hence it is proportional to the size of the population.

Letting $\Delta t \longrightarrow 0$ (and hence continuous time), the instantaneous rate of change becomes

$$\lim_{\Delta t \longrightarrow 0} \frac{\Delta P}{\Delta t} = \frac{dP}{dt} = kP$$

with $P(0) = P_0,\ 0 \le t \le t_1$.

**Solution:** Separating the variables, we have $\frac{1}{P}\,dP = k\,dt$.

Integrating, $\ln P = k\,t + \ln C,$ where $C$ is an arbitrary constant. Solving this, we get

$$P(t) = C\,e^{kt}.$$

Using the initial condition $P(0) = P_0$, we get $P_0 = C$. Hence the population at any time $t$ is

$$P(t) = P_0\,e^{kt}$$

**Remark 1:** This equation is known as *Malthusian Model* of Population Growth.

**Interpretation:**

i. The Proportionality constant $k = b - d$ is called the net growth rate or growth constant.

ii. If $k > 0$, then the population grows exponentially.

iii. If $k < 0$, then the population decays exponentially.

iv. if $k = 0$, then the population is constant. That is the population does not change over the time duration.

**Example 2.1** Interpret the solution graphically according as $k \gtreqless 0$.

**Remark.**

i. If $k > 0$, the population becomes double its present size at time $T$, called ***doubling period***, where

$$2P_0 = P_0\,e^{kT}$$

from which we get

$$T = \frac{1}{k}\ln 2 = \frac{1}{k}(0.69314).$$

ii. If $k < 0$, the population becomes half of its present size at time $T'$, called ***half-life period***, where

$$\frac{1}{2}P_0 = P_0\,e^{kT}$$

from which we get

$$T = \frac{1}{k}\ln\left(\frac{1}{2}\right) = -\frac{1}{k}(0.69314).$$

Note that both doubling period and half-life period are independent of $P_0$ and depend only on the net growth rate $k$.

**Example 2.2** According to an excerpt from a news article published in 1975, there are 21million babies born and 8 million deaths in a year in India. The population nearly 570 million is expected to reach a billion by the end of the century.

(i) Find the growth constant.

(ii) Assuming natural growth, estimate the population of India in the year 2000.

**Example 2.3** The initial bacterium count in a culture is 500.A biologist later makes a sample count of bacteria in the culture and finds that growth rate is 40% per hour.

(i) Find a model to find the number of bacteria after $t$ hours, $P(t)$ .

(ii) What is the estimated counter after one hour?

(iii) What is the count after ten hours?

(iv) Sketch the graph of function $P(t)$.

**Example 2.4** The 1990 Census for the population of the united states was 248,710,000. And in 1970 it was 203,211,926. Find the growth constant for the model and predict the population for 2000. The 2000 census for the United States was 281,400,000. Find the error of prediction in percentage. Estimate the population for 2050 and discuss the validity of your model.

**Remark 2:** Observe that the population grows exponentially and that predicted population for 2050 exceeds current estimates of the maximum sustainable population of the entire Earth. However. It is not happening as individual members eventually compete with food, living space and other natural resources. Thus, Malthusian model should be revised to reflect this competition which we will discuss later.

### 2.2.2 Effects of Immigration and Emigration on Population Size

If there is immigration into the population from outside at a rate proportional to the population size, the effect is equivalent to increasing birth rate. Similarly, if there is emigration from the population at a rate proportional to the population size, the effect is the same as that of increase in the death rate. If immigration and emigration take place at constant rate $i$ and $e$ respectively, The Malthusian model can be modified to

$$\begin{aligned}\frac{dP}{dt} &= kP + i - e \\ &= kP + \alpha \\ &= k\left(P + \frac{\alpha}{k}\right).\end{aligned}$$

Separating the variables as $\frac{1}{P+\frac{\alpha}{k}}dP = k\,dt$ and integrating, we get

$$\ln\left(P + \frac{\alpha}{k}\right) = kt + \ln C\,,$$

where $C$ is an arbitrary constant. This implies

$$P(t) + \frac{\alpha}{k} = Ce^{kt}$$

Imposing the initial condition $P(0) = P_0$, we obtain $C = P_0 + \frac{\alpha}{k}$. Thus,

$$P(t) = \left(P_0 + \frac{\alpha}{k}\right) e^{kt} - \frac{\alpha}{k}.$$

**Example 2.5** Suppose that there are 1000 birds on an island, breeding with a constant continuous growth rate of 10% per year. But now birds migrate to the island at a constant rate of 100 new arrivals per year. To three significant figures, how many birds are on the island after seven years?

## 2.3 Linear Growth Models in Physical Sciences

### 2.3.1 Radio-active Decay Model

**Problem Identification:** Many substances undergo a radioactive decay. Each of them has a half-life period. In radiology, this can be used in dead plants and animals to estimate the time of death. Radioactive dating can also be used to estimate the age of the solar-system and of Earth.

**Problem Simplification:** For a simple model, we may assume that the rate of radioactive decay is proportional to the amount of the radioactive substance present at any time.
In literature, we may able to find half-life period of many substances. For example, Uranium238 has half-life period 4.55 billion years whereas Carbon14 has 5568 years.

**Factors:** Let
$t$ be the time duration (independent variable),
$P(t)$ be the amount of the substance at time $t$ (dependent variable),
$T$ be the half-life period of the substance (constant),
$P_0$ be the initial amount of the substance (constant).

**Formulation:** By the assumption, we have

$$\frac{dP}{dt} \propto P \quad \text{or} \quad \frac{dP}{dt} = kP,$$

where $k$ is the proportionality constant.

**Solution:** Solving this, we get

$$P(t) = C\, e^{kt},$$

where $C$ is an arbitrary constant. If $P(0)$ be the initial amount of the substant, we have

$$P(0) = P_0 = C.$$

Thus,

$$P(t) = P_0\, e^{kt}$$

If the half-life of the substance is known, say it is $T$,

$$\frac{P_0}{2} = P_0\, e^{kT.}$$

This gives us

$e^{kT} = \frac{1}{2}$ or $kT = \ln 0.5$ or $k = -\frac{1}{T}\,(0.69315)$..

Thus, the solution is

$$P(t) = P_0 \exp\left(-0.69315\, \left(\frac{t}{T}\right)\right).$$

**Interpretation:** Observe that since $k < 0$, the population decays exponentially and for the substances with smaller half-life the population will vanish for large time.

**Example 2.6** It is observed that a sample contains 50g radium at the time of investigation. Half-life of radium is 1600 years. How long will it be until it contains 45g?

**Example 2.7** It is assumed that the corona virus cannot reproduce outside of the body. In a study of how long the coronavirus can last on different surfaces, the scientists recently found that the half-life of the novel coronavirus on cardboard is around four hours. Evaluate how many percentage of the virus inactivated after 24 hours.

### 2.3.2 A simple model of Temperature Flaw

**Problem Identification:** Suppose that we place an object in a surrounding medium whose temperature is different from the temperature of the object. Then there will be a heat flow from lower temperature to higher temperature. If the temperature of the object is higher than the that of environment, the body will get cool and conversely, when the temperature of the body is lower than that of environment, it will get warmed. One of interesting question arise here is when the temperature flow stops. That is, how the temperatures are brought into equilibrium?

**Problem simplification:** We assume that the situation is governed by the Newton's law of cooling. The law states that
The rate of change of temperature of a body is proportional to the difference between the temperature of the body and temperature of the surrounding medium.
We assume here that volume of the surrounding medium is large enough so that the heat of the object has a negligible effect on its surrounding temperature.

**Factors:** Let
$T(t)$ be the temperature of the object at any time $t$,
$T_{env}$ be the temperature of the surrounding medium, assumed to be constant,
$T_0$ be the initial temperature of the object (constant).

**Formulation:** By the newton's law of cooling, we have

$$\frac{dT}{dt} \propto (T - T_{env}) \quad \text{or} \quad \frac{dT}{dt} = k\,(T - T_{env}).$$

Solution: By separating the variables, we get

$$\frac{1}{T - T_{env}}\,dT = k\,dt.$$

Integrating, we obtain

$$\ln(T - T_{env}) = kt + \ln C,$$

where $C$ is an arbitrary constant. On simplification, we have

$$T(t) = T_{env} + C\,e^{kt}.$$

Using the initial condition $T(0) = T_0$, we have $C = T_0 - T_{env}$ and hence

$$T(t) = T_{env} + (T_0 - T_{env})\,e^{kt}.$$

**Interpretation:** When $T_0 > T_{env}$, the object get cools and in this case the value of $k$ will be negative as $\frac{dT}{dt} < 0$. After long time, $e^{kt} \longrightarrow 0$ and hence the temperature of the body will remain same as temperature of the environment. On the other hand, When $T_0 < T_{env}$, the object get warms and hence $\frac{dT}{dt} > 0$. In this case the value of $k$ will be positive.

**Example 2.8** Explain why ice cream pulled out from a refrigerator at $-4℃$ will get hotter more quickly than that pulled out from a refrigerator at $0℃$?

**Example 2.9** Police arrived at the scene of a murder at 12.30 a.m. They immediately take and record the body's temperature, which is $90°$F, and thoroughly inspect the area. By the time they finish the inspection, it is 2 a.m. They again take the temperature of the body which has dropped to $85°$F. The temperature at the crime scene has reminded steady at $80°F$. Find the time of murder.

## 2.4 Simple Linear Model in Mechanics

**Motion under Gravity in a Resisting Medium**

**Problem identification:** Consider the situation that a particle falls under gravity in a medium in which resistance is not negligible. We are interested with finding the distance travelled by a particle at a time before it hits the ground and the velocity at that point.

**Factors and Variables:**
$t -$ time taken for the displacement (independent variable)
$x(t) -$ displacement of a particle at time $t$ (dependent variable),
$v(t) -$ velocity at time $t$ (dependent variable),
$g -$ gravitational acceleration (constant),
$R(t) -$ resistance acceleration (dependent variable).

**Simplification:** Let us assume that the resistant is proportional to the velocity. Then,

$$R(t) \propto -mv \quad \text{or} \quad R = -kmv.$$

Velocity $v = \frac{dx}{dt}$.
Acceleration $a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v\frac{dv}{dt}$.

**Formulation:** By Newton's second law,

$$m\frac{dv}{dt} = mg - mkv \quad \text{or} \quad \frac{dv}{dt} = g - kv.$$

**Solution:**

$$\frac{dv}{dt} = -k\left(v - \frac{g}{k}\right).$$

Separating the variables, we have

$$\frac{1}{v - V}dv = -kdt, \qquad V = \frac{g}{k}.$$

Integrating, we obtain

$$\ln(v - V) = -kt + \ln C,$$

where $C$ is an arbitrary constant. On simplification, we get

$$v(t) = V + C\, e^{-kt}.$$

If the initial velocity $v(0) = v_0$, $C = v_0 - V$. Then,

$$v(t) = V + (v_0 - V)\, e^{-kt}.$$

To find the displacement $x(t)$, replace $v$ by $\frac{dx}{dt}$ and integrate:

$$\frac{dx}{dt} = V + (v_0 - V)\, e^{-kt}$$

and then

$$x(t) = Vt - \frac{(v_0 - V)}{k}\, e^{-kt} + A,$$

where $A$ is the integrating constant. Using $x(0) = 0$, we get

$$A = \frac{(v_0 - V)}{k}.$$

Therefore,

$$x(t) = Vt + \frac{(v_0 - V)}{k}(1 - e^{-kt}).$$

**Example 2.10** In the above model, assume that the particle falls from the rest. Find the velocity and displacement functions. What is the terminal speed? Draw the velocity – time graph and distance – time graph.

## 2.5 Linear Models in Finance

### Simple Linear Marketing Model

**Problem Identification:** Prize of a commodity changes with time in a market. Here we are interested to study how the prize changes in the market.

**Factors:**

$t -$ time (independent variable),
$P(t) -$ prize of the commodity at time $t$ (dependent variable),
$Q_d(t) -$ quantity demanded for the commodity at time $t$ (dependent variable),
$Q_s(t) -$ quantity supplied of the commodity at time $t$ (dependent variable).

**Assumptions:**

1. rate of change of the prize is proportional to the difference between the demand and supply:
$$\frac{dP(t)}{dt} \propto \big(Q_d(t) - Q_s(t)\big) \quad \text{or} \quad \frac{dP(t)}{dt} = k\big(Q_d(t) - Q_s(t)\big).$$
Here $k > 0$, since if demand is higher than supply, the prize increases.

2. $Q_d(t)$ is a linear function of $P(t)$: $Q_d(t) = d_1 + d_2 P(t)$.
Here $d_2 < 0$ as when the prize increases, the demand decreases.

3. $Q_s(t)$ is a linear function of $P(t)$: $Q_s(t) = s_1 + s_2 P(t)$.
Here $s_2 > 0$ as when the prize increases, the supply decreases.

**Formulation:**

$$\begin{aligned} \frac{dP(t)}{dt} &= k[d_1 - s_1 + (d_2 - s_2)P(t)] \\ &= k[\alpha - \beta P(t)]; \;\; \alpha = d_1 - s_1, \;\; \beta = s_2 - d_2 > 0. \end{aligned}$$

**Solution:** At the equilibrium prize $P_e, \frac{dP}{dt} = 0$ and hence $\alpha - \beta P_e = 0$ $or$ $P_e = \frac{\alpha}{\beta}$.

On the other hand, at the equilibrium prize $P_e$, $Q_d(t) = Q_s(t)$

$$d_1 + d_2 P_e = s_1 + s_2 P_e$$

which gives

$$P_e = \frac{d_1 - s_1}{s_2 - d_2} = \frac{\alpha}{\beta}.$$

Thus, the equation can be written as

$$\frac{dP}{dt} = -\beta k(P - P_e) = -K(P - P_e).$$

Separating the variable and solving, we get $P(t) - P_e = C\, e^{-Kt}$.

If the initial prize of the commodity is $P(0) = P_0$, then $C = P_0 - P_e$ and hence

$$P(t) = P_e - (P_e - P_0)\, e^{-Kt}.$$

**Interpretation:** After long period of time, i.e. when $t \to \infty$, $P(t) \to P_e$. That is, the market prize of the commodity becomes stable.

**Example 2.11** Suppose the quantity supplied $Q_s$ and quantity demanded $Q_d$ of T- shirts at a Company are,

$$Q_d = 1000 - 25P$$
$$Q_s = -200 + 50P$$

(i) Graph these two functions.
(ii) Find the equilibrium price and equilibrium Quantities.
(iii) If the initial prize of a T-shirt is Rs. 800.00 and after 1 year it is Rs. 875.00, find market price at time $t$.
(iv) Hence find market price at time $t = 12$.

## 2.5 Non-Linear Growth Model

### Logistic Law of Population Growth

We have discussed in the remark 2 that Malthusian model should be revised to some competencies.

**Problem Identification:** Let us model a situation when birth rate and death rate are not constants.

**Factors:** same factors as Malthusian model.

**Assumptions:** As population increases,

1. due to overcrowding and limitation of resources, the birth rate $b$ decreases linearly and has the
2. the death rate $d$ increases with population size linearly and has the form $d = d_1 + d_2 P$, where $d_1, d_2 > 0$.

**Formulation:** The Malthusian model now takes the form

$$\frac{dP}{dt} = (b-d)P = [b_1 - d_1 - (b_2 + d_2)P]P = P(a - bP), \qquad a, b > 0.$$

**Solution:** Separating the variables and resolving in to partial factions, we have

$$\left[\frac{1}{P} + \frac{b}{a - bP}\right] dP = a\,dt.$$

Integrating this, $\ln P - \ln(a - bP) = at + \ln C$, where $C$ is an arbitrary constant. Simplifying, we have

$$\frac{P(t)}{a - bP(t)} = C\,e^{at}.$$

Using the initial condition $P(0) = P_0$, we have $C = \frac{P_0}{a - bP_0}$. Thus,

$$\frac{P(t)}{a - bP(t)} = \frac{P_0}{a - bP_0}\,e^{at}.$$

$\implies \quad (a - bP_0)P(t) = P_0\big(a - bP(t)\big)e^{at}.$

$\implies \quad (a - bP_0 + bP_0e^{at})P(t) = aP_0e^{at}.$

$\implies \quad P(t) = \dfrac{aP_0e^{at}}{(a - bP_0 + bP_0e^{at})} = \dfrac{MP_0}{[P_0 + (M - P_0)e^{-at}]}; \;\; M = \dfrac{a}{b}.$

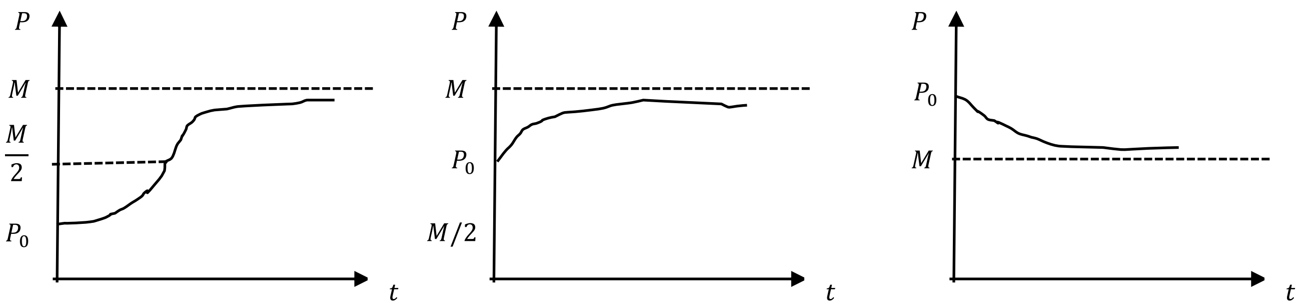
**Interpretation:**

(i) Equilibrium population $P_e$ is obtained letting $t \longrightarrow \infty$. That is,
$P_e = M = \dfrac{a}{b} = \dfrac{b_1 - d_1}{b_2 + d_2}.$

(ii) If $P_0 < M$, then $\frac{P_0}{[P_0 + (M - P_0)e^{-at}]} < 1$. This implies $P(t) < M$ and hence $\frac{dP}{dt} = P(a - bP) = bP(M - P) > 0$. Therefore, $P(t)$ is a monotonic increasing function of $t$ which approaches $M$ as $t \to \infty$.

(iii) If $P_0 > M$, then $P(t) > M$ and hence $\frac{dP}{dt} < 0$. Therefore, $P(t)$ is a monotonic decreasing function of $t$ which approaches $M$ as $t \to \infty$.

Further, $\frac{d^2P}{dt^2} = (a - 2bP)\frac{dP}{dt} \gtreqless 0$ according as $P \gtreqless \frac{a}{2b} = \frac{M}{2}$. That is, in case (ii), the growth curve is convex if $P < \frac{M}{2}$, concave if $P > \frac{M}{2}$ and has a point of inflexion if $P = \frac{M}{2}$.




$P_0 < M/2$ $\qquad$ $\frac{M}{2} < P_0 < M$ $\qquad$ $P_0 > M$

**Example 2.12** The population of the US in 1800 and 1850 was 5.3 and 23.1 million people respectively. Predict its population in 1900 and in 1950 using the exponential model of population growth. Then considering that the population of the US in 1900 was actually 76 million people correct your prediction for 1950 using the logistic model of population growth (help: with this data $k$ = 0.031476 in the logistic model). What is the carrying capacity of the US according to this model?