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DEPARTMENT OF MATHEMATICAL SCIENCES
FACULTY OF APPLIED SCIENCES

MTM 21012 MATHEMATICAL MODELING

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3. MATHEMATICAL MODELING THROUGH FIRST ORDER DIFFERENCE EQUATIONS

3.1 Introduction

In this chapter, we build Mathematical Models to describe change in an observed behavior.

Consider a sequence of numbers $\{a_0, a_1, a_2, a_3, \cdots, a_{n-1}, a_n, \cdots\}$. The *first differences* are

$$\Delta a_0 = a_1 - a_0,$$

$$\Delta a_1 = a_2 - a_1,$$

$$\Delta a_2 = a_3 - a_2,$$

and so on. For a positive integer n, the n^{th} first order difference is

$$\Delta a_n = a_{n+1} - a_n.$$

The first different represents the *change* of the sequence during one time period (raise or fall between consecutive values of the sequence).

An equation which expresses a value of a sequence as a function of the other terms in the sequence is called a *difference equation*. In particular, an equation which expresses the value a_n an of a sequence $\{a_n\}$ as a function of the term a_{n-1} is called a first-order difference equation. By solving a difference equation, we mean finding a function f such that $a_n = f(n), n = 1, 2, 3, \cdots$.

3.2 Linear Discrete Model in Biology

Example 3.1 Suppose a certain population of owls is growing at the rate of 2% per year. Let X_0 represent the size of the initial population of owls and X_n the number of owls n years later.

Find X_4 in terms of X_0.

Develop a discrete model to represent the population growth of owls.

Let there were 100 owls initially. Estimate the population of owls after (i) 20 years and (ii) 150 years.

In general, if the change occur at a constant rate r, then the model is

$$X_{n+1} = X_n + rX_n = (1 + r)\, X_n = \alpha X_n.$$

The general solution of which is

$$X_{n+1} = \alpha X_n = \alpha(\alpha X_{n-1}) = \alpha^2(\alpha X_{n-2}) = \cdots = \alpha^n X_0.$$

3.2 Effects of Immigration and Emigration on Population Size

In this case we assume that there is an external factor (immigration or emigration, in case of population change) influence the change of population. The situation can be modeled by

$$X_{n+1} = \alpha X_n + \beta.$$

If population is immigrated, then $\beta > 0$ and if emigrated, then $\beta < 0$.

Solving Procedure:

$$\begin{aligned}
X_n &= \alpha X_{n-1} + \beta \\
&= \alpha(\alpha X_{n-2} + \beta) + \beta = \alpha^2 X_{n-2} + \beta(\alpha + 1) \\
&= \alpha^2(\alpha X_{n-3} + \beta) + \beta(\alpha + 1) = \alpha^3 X_{n-3} + \beta(\alpha^2 + \alpha + 1) \\
&\ \ \vdots \\
&= \alpha^n X_0 + \beta(\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^2 + \alpha + 1).
\end{aligned}$$

Note that

$$\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^2 + \alpha + 1 = \frac{1-\alpha^n}{1-\alpha}; \ \ \alpha \neq 1.$$

Hence

$$X_n = \alpha^n X_0 + \beta\left(\frac{1-\alpha^n}{1-\alpha}\right).$$

When $\alpha = 1$,

$$X_n = X_0 + n\beta.$$

Example 3.2 A lake contains 10,000 fish at present. If there was no fishing the population of fish would increased by 15% every year. It is proposed to allow fishing at the rate 2000 fish per year. Model the fish population at the lake and interpret your answer,

What happen to the fish population eventually?

2.3 Linear Growth Models in Physical Sciences

3.3.1 Discrete Radio-active Decay Model

Example 3.3 Radium is a radioactive element which decays at a rate of 1% every 25 years. Assuming initial amount of radium exists on a body is 500 grams, Find the amount of radium left after 100 years. Express it as the percentage. Sketch a plot of the amount of radium left versus number of years. Find the half-life of radium.

2.3.2 A simple Discrete model of Temperature Flaw

If T_0 represents the initial temperature of the object, T_{env} the constant temperature of the surrounding environment, and T_n the temperature of the object after n units of time, then the Newton's low of cooling states that the change in temperature over a fixed unit of time is proportional to the difference between the temperature of the object and the temperature of the surrounding environment. Hence we can formulate the situation as

$$T_{n+1} - T_n = k(T_n - T_{env}),$$

where $n = 0, 1, 2, \cdots$ and k is a constant which depends upon the object.

Remark 3.1 This implies that large temperature differences result in a faster rate of cooling (or warming) than do small temperature differences.

Example 3.4 Suppose a cup of tea, initially at a temperature of 180°F , is placed in a room which is held at a constant temperature of 80°F. Moreover, suppose that after one minute the tea has cooled to 175°F. What will the temperature be after 20 minutes? What happen to the tea temperature eventually? Sketch the graph for tea temperature decreas.

3.4 Linear Discrete Models in Finance

3.4.1 Simple Interest Models

Suppose that we want to find the interest and total amount after n^{th} period. Let P represents the Principal (Present Value) and r the rate of interest per period.

The interest for each period will be Pr. Thus,
Total interest after n periods will be $I_n = nPr$.
Total amount after n periods is $P_n = P + nPr = P(1 + nr)$.

P_n is called the future value or accumulated value.

Example 3.5 Sehas invest \$6000 on saving account of bank that pay Simple interest at a rate 8% per annum. If the interest is paid at the end of each month, how much interest Sehas get after 5 months. How long it will take double Sehas Money.

Example 3.6 Calculate the Future Value after 5 years of an investment \$ 1000 made now for the following interest rate scenarios:
At Simple rate of interest of 6% per annum for first 3 years and reinvested at simple rate of interest of 7% per annum for next 2 years. The interest is paid at the end of each year.

3.4.2 Compound Interest Models

Compound interest (or compounding interest) is the interest on a loan or deposit calculated based on both the initial principal and the accumulated interest from previous periods. The rate at which compound interest accrues depends on the frequency of compounding, such that the higher the number of compounding periods, the greater the compound interest.

Let P be the initial amount deposited at a constant interest rate r compounded at the end of each period. The principle value for a current term will be the principle value of the previous term plus interest. Thus,

At the end of the first term, the future value is $P_1 = P + Pr = (1 + r)P$.

This will be the principle value for the second term. The interest will be paid for this amount. Therefore, at the end of the second term the future value becomes $P_2 = (1 + r)P + (1 + r)Pr = (1 + r)^2 P$.

Similarly, at the end of the third year, the future value is $P_3 = (1+r)^2P + (1+r)^2Pr = (1+r)^3P$. Continuing in the same way

$$P_n = (1+r)^nP.$$

Exercise: Find a formular for the total compound interest I_n after n terms.
Hint: $I_n = (1+r)^nP - P = P[(1+r)^n - 1]$.

Example 3.7

a. Find the future value of \$ 5000 after 8 years under compound, the principle value at the end of n^{th} year will be interest rate 8%.
b. Find the present value of \$ 12 500 of 06$^{\text{th}}$ year under 7%
c. What interest rate doubles your investment amount end of 5$^{\text{th}}$ year?
d. How long you need to wait double your money under 10% rate of interest.

Remark 2:

- In reality, interest means: Compounding interest otherwise need to specify it.
- Usually, annual interest and annual compounding otherwise need to specify.

Compounding Frequency: If the interest can be compounded more than one periods per year, say m number of period per year, then the principle value at the end of t^{th} year will be

$$P_n = P\left(1+\frac{r}{m}\right)^n = P\left(1+\frac{r}{m}\right)^{mt}.$$

Remarks:

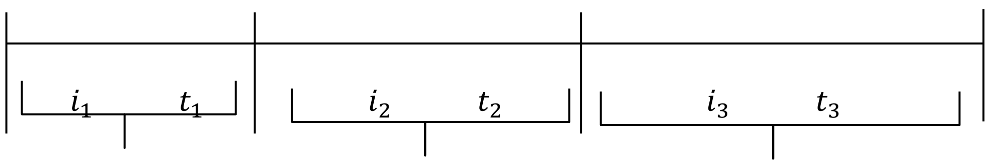
m = 1 annual
m = 2 Semi Annual or bi annual
m = 4 Quarterly
m = 12 Monthly
m = 52 Weekly
m = 365 Daily

Example 3.8 An investor deposits \$2,000 in to an account which offers 3% quarterly rate of interest and compounded quarterly. Compute the future value after 5 years.

Example 3.9 Sehas borrows \$ 12,000 and agree to settle it after one month by paying \$ 12,250.Compute the annual rate interest.

3.4.3 Varying rate of interest

When a variable interest rate moves up and down based on another interest rate:



Future value of first period can be considered as present value of second period.

$$A = P(1+i_1)^{t_1}(1+i_2)^{t_2}(1+i_3)^{t_3}$$

Example 3.10 Find the accumulated value of \$100 at the end of 15 years if the effective rate of interest is 5% for the first 5 years, 4.5% for second five years and 4% for the third five years.