SOUTH EASTERN UNIVERSITY OF SRI LANKA

MTS 00033 MULTIVARIATE CALCULUS

ASSIGNMENT 1

Limits and Continuity

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1. Find the natural domain and range of the function $f(x,y) = \frac{1}{\sqrt{4-x^2-y^2}}$.

2. Find the domain of $f(x,y) = 8\ln(2 + x + y^2)$.

3. Find the natural domain and range of the function $f(x,y) = \sqrt{x^2 + y^2 - 9}$.

4. Using ϵ, δ definition, prove each of the following:

   (a) $\lim\limits_{(x,y)\to(1,2)}(3xy) = 6,$ (b) $\lim\limits_{(x,y)\to(1,1)}\left(\frac{x^2-1}{3x+y}\right) = 1.$

5. Prove that if the limit $\lim\limits_{(x,y)\to(a,b)}[f(x,y)]$ exists, then it is unique.

6. Evaluate each of the following limit or explain why it fails to exist:

   (a) $\lim\limits_{(x,y)\to(0,0)}\left(\frac{x^2\sin^2 y}{x^2+2y^2}\right),$ (b) $\lim\limits_{(x,y)\to(0,0)}\left(\frac{x^2}{x^2+y^2}\right),$

   (c) $\lim\limits_{(x,y)\to(0,0)}\left(\frac{x^3+xy^2+2x^2+2y^2}{x^2+y^2}\right),$ (d) $\lim\limits_{(x,y)\to(0,0)}\left[\frac{x^2y^2}{x^2+y^2}\right],$

   (e) $\lim\limits_{(x,y)\to(0,0)}\left(\frac{x\sin x}{x^2+y^4}\right),$ (f) $\lim\limits_{(x,y)\to(0,0)}\left(\frac{x\sqrt{|y|}\sin x}{x^2+y^4}\right),$

   (g) $\lim\limits_{(x,y)\to(0,0)}\left[\frac{12x^3y^5+4x^4y^4}{x^6+4y^8}\right],$ (h) $\lim\limits_{(x,y)\to(0,0)}\left[\frac{12x^3y^4+4x^4y^4}{x^6+4y^8}\right],$

   (i) $\lim\limits_{(x,y)\to(0,0)}\left[\frac{\sin(x^2+y^2)}{x^2+y^2}\right],$ (j) $\lim\limits_{(x,y)\to(0,0)}\left[\frac{\sin(xy)}{\sqrt{x^2+y^2}}\right],$

   (k) $\lim\limits_{(x,y)\to(0,0)}\left[\frac{\sin(xy)}{x^2+y^2}\right],$ (l) $\lim\limits_{(x,y)\to(0,0)}\left[\frac{x^3-2y^3}{2x^2+3y^2}\right],$

   (m) $\lim\limits_{(x,y)\to(0,0)}\left[\frac{|x|+|y|}{x^2+5y^2}\right],$ (n) $\lim\limits_{(x,y)\to(0,0)}\left[\frac{\sin(xy)}{\sin x\sin y}\right],$

   (o) $\lim\limits_{(x,y)\to(0,0)}\left[\frac{\sin(x^2+y^2)}{1-\cos\sqrt{x^2+y^2}}\right],$ (p) $\lim\limits_{(x,y)\to(0,0)}\left[\frac{x^2y^2}{x^2y^2+(x-y)^2}\right],$

   (s) $\lim\limits_{(x,y)\to(0,0)}\left[\frac{|x|^p\,|y|^q}{x^2+y^2}\right],$ (t) $\lim\limits_{(x,y)\to(0,0)}\left[\frac{|x|^p\,|y|^q}{(x^2+y^2)^{3/2)}}\right].$

7. Evaluate each of the following limits or show that the limit does not exist:

   (a) $\lim\limits_{(x,y,z)\to(0,0,0)}\left[\frac{(x+y+z)^2}{x^2+y^2+z^2}\right],$ (b) $\lim\limits_{(x,y,z)\to(1,0,0)}\left[\frac{(x+y+z)}{e^{x^2+y^2+z^2}}\right],$ (c) $\lim\limits_{(x,y,z,t)\to(0,0)}\left[\frac{xyz}{x^2+y^2+z^2}\right].$

8. Evaluate each of the following repeated limits:

$(a)\ \lim_{x\to 0}\left[\lim_{y\to 0}\left(\frac{x^3-y^3}{x^3+y^3}\right)\right]$ and $(b)\ \lim_{y\to 0}\left[\lim_{x\to 0}\left(\frac{x^3-y^3}{x^3+y^3}\right)\right]$,

Compare your result with

$$\lim_{(x,y)\to(0,0)}\left[\frac{x^3-y^3}{x^3+y^3}\right].$$

9. Evaluate each of the following repeated limits:

$(a)\ \lim_{x\to 0}\left[\lim_{y\to 0}\left(\frac{2x^2y}{x^4+y^2}\right)\right]$ and $(b)\ \lim_{y\to 0}\left[\lim_{x\to 0}\left(\frac{2x^2y}{x^4+y^2}\right)\right]$,

Compare your result with

$$\lim_{(x,y)\to(0,0)}\left[\frac{x^3-y^3}{x^3+y^3}\right].$$

10. Let $f(x,y)=\begin{cases} xy\sin\left(\frac{1}{x^2+y^2}\right); & (x,y)\neq(0,0) \\ 0; & (x,y)=(0,0)\end{cases}$. Evaluate

$(a)\ \lim_{x\to 0}\left[\lim_{y\to 0}(f(x,y))\right]$, $(b)\ \lim_{y\to 0}\left[\lim_{x\to 0}\left(\frac{xy}{x^2+y^2}\right)\right]$ and $(c)\ \lim_{(x,y)\to(0,0)}[f(x,y)]$.

11. Show that the function $f(x,y)=\begin{cases} 3xy; & (x,y)\neq(2,3) \\ 6; & (x,y)\neq(2,3)\end{cases}$ has a discontinuity at $(2,3)$.

Suitably redefine the function f to make it continuous.

12. Does the function $f(x,y)=\frac{3x^2y}{\sin\pi x}$ has any discontinuities? Justify your answer.

13. Show that the function $f(x,y)=\begin{cases} \frac{2xy^2}{x^3+y^3}; (x,y)\neq(0,0) \\ 0; \quad (x,y)=(0,0)\end{cases}$ is discontinuous at $(x,y)=(0,0)$.

14. Investigate the continuity of the function $f(x,y)=\begin{cases} \frac{x^2y^4}{(x^2+y^4)^2}; & (x,y)\neq(0,0) \\ 0; & (x,y)=(0,0)\end{cases}$.

15. Investigate the continuity of the function $f(x,y)=\begin{cases} \frac{xy(x^2-y^2)}{x+y}; & (x,y)\neq(0,0) \\ 0; & (x,y)=(0,0)\end{cases}$.

16. Investigate the continuity of the function $f(x,y)=\frac{x^2+x^3y}{x+y}$.

17. Let $f(x,y)=\frac{3x^2y}{x^2+y^2}$. Where is f continuous?

18. Examine the continuity of $\lim_{(x,y)\to(0,0)}\left[(x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)\right]$.

19. Let $f(x,y)=\ln\left(\frac{3x^2-2x^2y^2+3y^2}{2x^2+2y^2}\right)$; $(x,y)\neq(0,0)$. Define $f(0,0)$ in such a way that f to be continuous at the origin.