SOUTH EASTERN UNIVERSITY OF SRI LANKA
DEPARTMENT OF MATHEMATICAL SCIENCES
FACULTY OF APPLIED SCIENCES

MTM 21012 MATHEMATICAL MODELING

---

4. MATHEMATICAL MODELING THROUGH SECOND ORDER DIFFERENTIAL EQUATIONS

4.1 Introduction

A second order linear differential equation with constant coefficients has the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c = Q(x),$$

where a, b, c are given constants with $a_2 \neq 0$.

The general solution of this equation is

$$y = \text{homogeneous solution} + \text{particular integral.}$$

Homogeneous solution:

Replace $\frac{d^r y}{dx^r}$; $0 \leq r \leq 2$ by λ^r and set $Q(x) = 0$. Then, solve the resulting auxiliary equation

$$a\,\lambda^2 + b\,\lambda + c = 0.$$

Solve this equation for λ.

Case 1 $\boldsymbol{b^2 - 4ac > 0}$. Both roots are real and distinct.
In this case, the homogeneous solution is

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x},$$

where C_1, C_2 are arbitrary constants.

Case 2 $\boldsymbol{b^2 - 4ac = 0}$. Both are real but repeated.
In this case, the homogeneous solution is

$$y = (C_1 + C_2 x)\, e^{\lambda x},$$

where C_1, C_2 are arbitrary constants.

Case 3 $\boldsymbol{b^2 - 4ac < 0}$. Both roots are complex conjugates of the form $\boldsymbol{\lambda = \alpha \pm i\beta}$.
In this case, the homogeneous solution is

$$y = e^{\alpha x}[C_1 \cos\beta x + C_2 \sin\beta x],$$

where C_1, C_2 are arbitrary constants.

Particular Solution:

Now consider the equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c = Q(x)$. We shall discuss finding particular integral $y_p(x)$ only for certain class of the function $Q(x)$ by the method of undetermined coefficients.

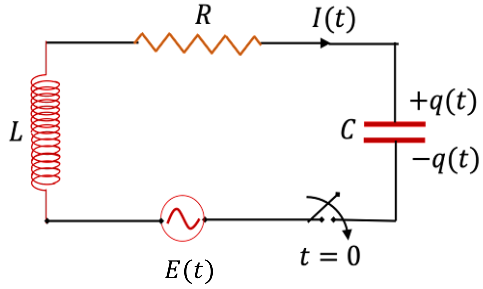
$Q(x)$	$y_p(x)$ guess
$ae^{\alpha x}$	$Ae^{\alpha x}$
$a\cos\alpha x$	$A\cos\alpha x + B\sin\alpha x$
$a\sin\alpha x$	$A\cos\alpha x + B\sin\alpha x$
$a\cos\alpha x + b\sin\alpha x$	$A\cos\alpha x + B\sin\alpha x$
n^{th} degree polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$	n^{th} degree polynomial $A_0 + A_1x + A_2x^2 + \cdots + A_nx^n$

Remark: If any term of y_p is a solution of the complementary equation, multiply y_p by x (or a suitable power of x if necessary).

4.2 A Model in Physical Sciences

Background:

Consider the situation that a inductor (coil) L, a resistor R, a capacitor C, a source of electricity (Battery / Generator) and a switch are connected in series in a closed circuit. The electric source produces a voltage $E(t)$ volts.



Factors and variables:

Inductance L Henrys
Resistance R Ohms
Current I Ampere
Charge q coulomb / faraday
Time t seconds
Voltage E watt
Electro motive fore E watt

Simplification: Definitions and Physical laws

1. Current $I(t)$ amperes is the time rate of change of charge q coulombs. i.e.

$$I(t) = \frac{dq}{dt}.$$

2. The potential drop due to resistance R ohms is

$$E_R = IR = R\frac{dq}{dt}.$$

3. The potential drop due to inductance L henrys is

$$E_L = L\frac{dI}{dt} = L\frac{d^2q}{dt^2}.$$

4. The potential drop due to Capacitance C coulombs is

$$E_C = \frac{q}{c}.$$

5. Kirchhoff's Law: The algebraic sum of the voltage drops across a closed circuit is zero. i.e.

$$E_L + E_R + E_C - E = 0.$$

This gives

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = E(t),$$

which is a second order linear non-homogeneous differential equation with constant coefficients.

Example 4.1 An inductor of 2 henrys, resistor of 16 ohms, and a capacitor of 0.02 farads are connected in a series with an electro motive force E volts. At $t = 0$, the charge on the capacitor and the current on the circuit are zero. Find the charge and the current at any time $t > 0$ if

(a) $E = 300$ volts,

(b) $E = 100\sin 2t$ volts.