Example 4.1 An inductor of 2 henrys, resistor of 16 ohms, and a capacitor of 0.02 farads are connected in a series with an electro motive force E volts. At $t = 0$, the charge on the capacitor and the current on the circuit are zero. Find the charge and the current at any time $t > 0$ if
(a) $E = 300$ volts,
(b) $E = 100 \sin 2t$ volts.

Solution: The governing equation is

$$2\frac{d^2q}{dt^2} + 16\frac{dq}{dt} + \frac{q}{0.02} = E(t) \quad \text{or} \quad 2\frac{d^2q}{dt^2} + 16\frac{dq}{dt} + 50q = E(t) - - - - - (1)$$

Auxiliary equation is

$$2\lambda^2 + 16\lambda + 50 = 0 \quad \text{or} \quad \lambda^2 + 8\lambda + 25 = 0.$$

Roots of the auxiliary equations are

$$\lambda = \frac{-8 \pm \sqrt{64 - 100}}{2} = -4 \pm 3i.$$

Thus, the homogeneous solution is

$$q_h(t) = e^{-4t}[C_1 \cos 3t + C_2 \sin 3t],$$

where C_1, C_2 are arbitrary constants.

When $E = 100 \sin 2t$ volts, particular integral is of the form

$$q_p(t) = A \cos 2t + B \sin 2t.$$

This gives

$$\frac{dq}{dt} = -2A \sin 2t + 2B \cos 2t$$

and

$$\frac{d^2q}{dt^2} = -4A \cos 2t - 4B \sin 2t.$$

Substituting these in (1), we obtain
$2(-4A \cos 2t - 4B \sin 2t) + 16\,(-2A \sin 2t + 2B \cos 2t) + 50\,(A \cos 2t + B \sin 2t) =$
$100 \sin 2t.$

$$\Longrightarrow \quad (42A + 32B) \cos 2t + (-32A + 42B) \sin 2t = 100 \sin 2\,t$$

Equating the coefficients, we have

$$21A + 16B = 0 \quad \text{and} \quad -16A + 21B = 50.$$

Solving these, we obtain

$$A = -\frac{800}{697}, B = \frac{1050}{697}.$$

Hence the general solution is,

$$q(t) = e^{-4t}[C_1 \cos 3t + C_2 \sin 3t] - \frac{800}{697}\cos 2t + \frac{1050}{697}\sin 2t.$$

Differentiating with respect to t, we get

$$I(t) = \frac{dq}{dt} = e^{-4t}[-(3C_1 + 4C_2) \sin 3t - (4C_1 - 3C_2) \cos 3t] + \frac{1600}{697}\sin 2t + \frac{2100}{697}\cos 2t.$$

Using the initial conditions $q(0) = 0$, we have

$$C_1 - \frac{800}{697} = 0 \quad \text{or} \quad C_1 = \frac{800}{697}.$$

Using the initial conditions $I(0) = 0$, we have

$$-4C_1 + 3C_2 + \frac{2100}{697} = 0 \quad \text{or} \quad C_2 = \frac{1100}{2091}.$$

Thus,

The charge

$$q(t) = \frac{100}{2091} e^{-4t}[24\cos 3t + 11\sin 3t] - \frac{800}{697}\cos 2t + \frac{1050}{697}\sin 2t$$

and the current

$$I(t) = -\frac{100}{2091} e^{-4t}[63\cos 3t + 116\sin 3t] + \frac{1600}{697}\sin 2t + \frac{2100}{697}\cos 2t.$$