2 PARTIAL DIFFERENTIATIONS

2.1 Partial Derivatives

Definition 2.1 Let $f(x_1, x_2, x_3, \dots\dots x_n)$ be a function of n variables. The derivatives of f with respect to the variable x_i, $1 \le i \le n$, when all others are kept constant is called the partial derivative of f with respect to x_i and is denoted by $\frac{\partial f(\underline{x})}{\partial x_i}$ or $f_{x_i}(\underline{x})$. Simply $\frac{\partial f}{\partial x_i}$ or f_{x_i}. That is,

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, x_2, \cdots, x_{i+h}, \cdots, x_n) - f(x_1, x_2, \cdots, x_i, \cdots, x_n)}{h}.$$

Example 2.1 Let $f(x,y) = xy$. Using the definition, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Example 2.2 Let $f(x,y) = e^x \sin y$. Using the definition, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Example 2.3 Suppose that $f(x,y,z) = e^{xy} \ln z$. Find f_x, f_y, f_z and show that $xf_x = yf_y$.

Example 2.4 Let $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & ;(x,y) \neq (0,0) \\ 0 & ;(x,y) = (0,0) \end{cases}$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Remark 2.1 Recall that for a function of single variables

- If f' exists, then f is continuous. But the converse is not true. i.e. if the function is continuous, it need not to be differentiable. For example, $f(x) = |x|$ is continuous at the origin, but differentiable there at.

- If f is not continuous, then f is not differentiable.

However, In the case of function of several variables, partial derivatives may exist though function is not continuous.

In the example 2.4, f is not continuous at origin but $f_x(0,0) = f_y(0,0) = 0$

2.2 Directional Derivatives

The partial derivative $f_x(x,y)$ gives the rate of change of f with respect to x (y constant). Similarly, $f_y(x,y)$ measures the rate of change of f with respect to y (x constant). If both x and y changes simultaneously, then how to find the rate of change?

Definition 2.2 The rate of change of the function $f(\underline{x})$ in the direction with the unit vector $\underline{\hat{u}} = \underline{a}$ is called the directional derivative and is denoted by $\mathcal{D}_{\underline{\hat{u}}} f(\underline{x})$. i.e.

$$\mathcal{D}_{\underline{\hat{u}}} f = \lim_{h \to 0} \frac{f(x_1 + a_1 h_1,\ x_2 + a_2 h_2, \dots, x_n + a_n h_n) - f(x_1,\ x_{2,\dots}\, x_n)}{h},$$

provided the limit exists.

Note: The directional derivative of $f(x,y)$ at the point (x_0, y_0) in the direction of positive X axis:
We know that $\underline{\hat{u}} = \underline{i} = (1,0)$. Thus

$$\mathcal{D}_{\underline{i}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h_1 , y_0) - f(x_0, y_0)}{h} = f_x(x_0, y_0).$$

Similarly, we get $\mathcal{D}_{\underline{j}} f(x_0, y_0) = f_y(x_0, y_0)$.

Theorem 2.1 Suppose that $f(\underline{x})$ is a function whose first order partial derivatives exist. The directional derivative of f in the direction of $\underline{\hat{u}} = \underline{a}$ is given by

$$\begin{aligned} \mathcal{D}_{\underline{\hat{u}}} f &= \frac{\partial f}{\partial x_1} a_1 + \frac{\partial f}{\partial x_2} a_2 + \cdots + \frac{\partial f}{\partial x_n} a_n \\ &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right) . (a_1, a_2, \ldots, a_n) \\ &= \nabla f . \underline{a} \end{aligned}$$

Note: $\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)$ is the gradient of the vector of the scalar function $f(\underline{x})$.

Example 2.5 Find the directional derivatives of $f(x,y) = xe^{xy} + y$ in the direction $\theta = \frac{2\pi}{3}$ at the point (2,3).

Theorem 2.2 Let $f(\underline{x})$ be a function whose first order partial derivatives exist and let P be a point in the domain of f. The maximum rate of change of function $f(\underline{x})$ is given by $\|\nabla f(\underline{x})\|$ and that will occur in the direction of $\nabla f(\underline{x})$

Proof: **Let** θ be the angle between ∇f and $\underline{\hat{u}}$

$$\begin{aligned} \mathcal{D}_{\underline{\hat{u}}} f &= \nabla f \cdot \underline{\hat{u}} \\ &= \|\nabla f\| \, \|\underline{\hat{u}}\| \cos\theta \\ &= \|\nabla f\| \cos\theta \end{aligned}$$

$\therefore$ $\mathcal{D}_{\underline{\hat{u}}} f$ is maximum when $\cos\theta = 1$ ($i.e.$ at $\theta = 0$).

Therefore, $(\mathcal{D}_{\underline{\hat{u}}} f)_{max} = \|\nabla f\|$.

Further, $\theta = 0 \implies \underline{\hat{u}}$ is in the direction of ∇f.

Example 2.6 Let $f(x,y) = xe^y$ defined on $\mathbb{R}^2$ and let $P \equiv (2,0)$, $Q \equiv \left(\frac{1}{2}, 2\right)$ be two points on $\mathbb{R}^2$.

(i) Find the rate of change of f at P in the direction of $\overrightarrow{PQ}$.

(ii) In which direction does f have the maximum rate of change and what is the value of maximum rate of change.

2.3 Tangent Planes of Level Surfaces

Let $f(x,y)$ be a function of two variable. The graph of f is a surface in $\mathbb{R}^3$ with the equation $z = f(x,y)$.

Let $\underline{r}(x,y) = (x,y,z) = (x,y,f)$ be a point on the surface. Then,

$$\frac{\partial \underline{r}}{\partial x} = (1, 0, f_x), \qquad \frac{\partial \underline{r}}{\partial y} = (0, 1, f_y).$$

The surface normal vector at the point $P_0(x_0, y_0, z_0)$ on the surface is

$$\underline{n} = \left[\frac{\partial \underline{r}}{\partial x} \times \frac{\partial \underline{r}}{\partial y}\right]_{(x_0,y_0)} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \left(-f_x(x_0, y_0), -f_y(x_0, y_0), 1\right).$$

Let $P(x, y, z)$ be an arbitrary point on the surface. Equation of tangent plane at $P_0(x_0, y_0, z_0)$ on the surface is

$\overrightarrow{PP_0} \cdot \underline{n} = 0.$

$$\Rightarrow \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} \cdot \begin{pmatrix} -f_x(x_0, y_0) \\ -f_y(x_0, y_0) \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow \quad -f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + (z - z_0) = 0$$

$$\Rightarrow \quad z = z_0 + (x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0).$$

Definition 2.3 Consider the implicit function $F(x, y, z) = C$, where C is an arbitrary constant. For each value of C, these represents a family of surfaces, called level surfaces.

We may take $F(x, y, z) - C = z - f(x, y) = 0$. Then,

$$\frac{\partial F}{\partial x} = -f_x, \qquad \frac{\partial F}{\partial y} = -f_y, \qquad \frac{\partial F}{\partial z} = 1.$$

Therefore, $\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = \left(-f_x, -f_y, 1\right).$

Hence the tangent plane for the level surface is

$$\begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} \cdot \nabla F = 0.$$

This shows that ∇F is a vector perpendicular to the tangent plane. The unit surface normal vector to the level surface $F(x, y, z) = C$ is $\frac{\nabla F}{\|\nabla F\|}$.

Example 2.7 Find the equation of the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Example 2.8 Find the equation of the tangent plane and surface normal to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$ at the point $(-2, 1, 3)$.

2.4 Higher Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(f_x) = f_{xx}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

Other higher order derivatives are similarly defined.

Example 2.9 Suppose that

$$f(x,y) = \begin{cases} xy\,\dfrac{(x^2 - y^2)}{(x^2 + y^2)} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}.$$

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Sufficient condition for the equality of f_{xy} and f_{yx}

Theorem 2.3 Let (a,b) be a point in the domain of the function of the function $f(x,y)$. If f_y exist in the neighborhood of (a,b) and f_{yx} is continuous at (a,b) then f_{xy} exist and equal to f_{yx}.

Remark: The condition stated in the theorem 2.3 are sufficient but not necessary. i.e. If the conditions are not satisfied then f_{xy} , f_{yx} may or may not equal.

Example 2.10 Let

$$f(x,y) = \begin{cases} \dfrac{(x^2\,y^2)}{x^2 + y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}.$$

Show that $f_{xy}(0,0) = f_{yx}(0,0)$.

Solution: Home work.

During the process, we show that $f_y(0,0) = 0$.
To show f_{yx} is not continuous at the origin at $(x,y) \neq (0,0)$.

$$f_y = \frac{(x^2 + y^2).\,2xy^2 - x^2\,y^2.\,2x}{(x^2 + y^2)^2} = \frac{2xy^4}{(x^2 + y^2)^2}$$

$$f_y = \begin{cases} \dfrac{2x\,y^4)}{(x^2 + y^2)^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

$$f_{xy} = \frac{8x^3y^3}{(x^2 + y^2)^3} \qquad ; (x,y) \neq (0,0)$$

Along the x – axis,

$$\lim_{x \to 0} f_{yx}(x,0) = 0.$$

Along the line $y = x$,

$$\lim_{(x,y)\to(0,0)} f_{yx}(x,y) = \lim_{(x,y)\to(0,0)} \frac{8x^6}{8x^6} = 1.$$

Therefore, $\lim_{(x,y)\to(0,0)} f_{yx}$ does not exist which gives

$$\lim_{(x,y)\to(0,0)} f_{yx} \neq f_{yx}(0,0).$$

Hence f_{yx} is not continuous.

- We proved $f_{xy}(0,0) = f_{yx}(0,0)$, but condition is not satisfied.