3 TOTAL DIFFERENTIATION

3.1 Linear Approximation

Since the tangent plane to the surface $z = f(x, y)$ at a point P on the surface is very closed to the surface at least when it is closed to P, we may use the function defining the tangent plane as a linear approximation to f.

Example 3.1 Find the equation of the tangent plane to the surface $z = 2x^2 + y^2$ at the point $P(1, 1, 3)$. Hence, estimate the values of $f(1.1, 0.95)$ and $f(23)$. Compare your estimations with the exact values in each case.

Definition 3.1 The linear approximation $L(x, y)$ to the surface $z = f(x, y)$ at the point $P(a, b)$ whose graph is the tangent plane to the surface at P is given by

$$L(x, y) = f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b).$$

L is called the linearization of f at (a, b) and the approximation $z = f(x, y) \approx L(x, y)$ is called linear approximation (or tangent plane approximation) of f at (a, b).

3.2 Differentiable Functions

Definition 3.2 Let (a, b) and $(a + h\,, b + k)$ be two nearby points of the domain D of a function $z = f(x, y)$. Then the change Δf of f as the the point (a, b) moves to the point $(a + h\,, b + k)$ is

$$\Delta z = \Delta f = f(a + h\,, b + k) - f(a, b).$$

The function $z = f(x, y)$ is said to be differentiable at (a, b) if there exist functions $\in_1 (h, k)$, $\in_2 (h, k)$ such that

$$\Delta z = f_x(a, b)h + f_y(a, b)\ k + \in_1 h + \in_2 k,$$

where $\lim\limits_{(h,k)\to(0,0)} \in_1 = 0$ and $\lim\limits_{(h,k)\to(0,0)} \in_2 = 0.$

If the function is differentiable at every point of D, then it is said to be differentiable on D.

Example 3.2 Prove that $f(x, y) = xy$ is differentiable on $\mathbb{R}^2$.

Theorem 3.1 If the function $f(x, y)$ is differentiable at (a, b), then it is continuous and possess first order partial derivatives at (a, b).
The converse is not necessarily true. i.e. if f is continuous at (a, b) and first order partial derivatives exist at (a, b) then, the function f may or may not differentiable there at.

Example 3.3 (Counter Example)

Show that $f(x, y) = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2} & ; (x, y) \neq 0 \\ 0 & ; (x, y) = 0 \end{cases}$ is continuous and possess first order partial derivatives, but not differentiable at the origin.

Theorem3.2 Sufficient condition for Differentiability

Suppose that $f_x(x, y)$ and $f_y(x, y)$ exist in an open neighborhood containing (a, b) and one of the partial derivative, say $f_y(x, y)$, is continuous at (a, b). Then, f is differentiable at (a, b).

Proof: Since $f_y(x, y)$ is continuous at (a, b), there exist an open ball B centered at (a, b) at every point of which f_y exists. Take any $(a + h, b + k) \in B$. Then,

$$f(a + h, b + k) - f(a, b) = \{f(a + h, b + k) - f(a + h, b)\} + \{f(a + h, b) - f(a, b)\}.$$

Now, consider the function of one variable

$$\phi(y) = f(a + h, y).$$

Since f_y exists in B, $\phi(y)$ is differentiable in the closed interval $[b, b + k]$ and hence we can apply Lagrange's Mean Value Theorem for $\phi(y)$. Thus, there exists a number θ, such that $0 < \theta < 1$ and

$$\phi(b + k) - \phi(b) = k\, \phi'(b + \theta k) = k f_y(a + h, b + \theta k).$$

Now, if we write

$$f_y(a + h, b + \theta k) - f_y(a, b) = \epsilon_2\ (h, k),$$

Then from the fact that f_y is continuous at (a, b), we obtain

$$\epsilon_2 \to 0 \ \text{ as } \ (h, k) \to (0,0).$$

Further, because f_x is exists at (a, b) implies

$$f(a + h, b) - f(a, b) = h\, f_x(a, b) + \epsilon_1\ (h, k)$$

where $\epsilon_1 \to 0$ as $(h, k) \to (0,0)$.

Combining these two, we get

$$\begin{aligned} f(a + h, b + k) - f(a, b) &= k\{f_y(a, b) + \epsilon_2\} + h\, f_x(a, b) + \epsilon_1 \\ &= f_x(a, b)h + f_y(a, b)\ k + \epsilon_1\ h + \epsilon_2\ k, \end{aligned}$$

where $\epsilon_1\ (h, k), \epsilon_2\ (h, k) \to 0$ as $(h, k) \to (0,0)$.

Thus, f is differentiable at (a, b).

Example 3.4 Show that $f(x, y) = \begin{cases} xy\left(\frac{x^2 - y^2}{x^2 + y^2}\right) & ;\ x^2 + y^2 \neq 0 \\ 0 & ;\ x = 0,\ y = 0 \end{cases}$ is differentiable at the origin.

Remark: The condition of continuity in the above theorem is sufficient but not necessary. That is, if the function is not continuous at a point (a, b), then f may or may not differentiable there at.

Example 3.5 Let

$$f(x, y) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + y^2 \sin\left(\frac{1}{y}\right) & ;\quad xy \neq 0 \\ x^2 \sin\left(\frac{1}{x}\right) & ; x \neq 0,\ y = 0 \\ y^2 \sin\left(\frac{1}{y}\right) & ; x = 0,\ y \neq 0 \\ 0 & ; x = 0,\ y = 0 \end{cases}$$

Show that $f_x(0,0), f_y(0,0)$ exit, both are discontinuity at the origin, but the function is differentiable.

3.3 The Differentials

Definition 3.3 Let $f(x_1, x_2, \cdots, x_n)$ be a differentiable function of n variables. The differential (total derivative) df is defined by

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n.$$

Remark: Consider the function $z = f(x, y)$. The total differential

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

is an estimate for the actual change (with the difference in the linear (tangent plane) approximation)

$$\Delta z = f(x + h, y + k) - f(x, y)$$

in response to (small) changes dx and dy in the input variables.

Example 3.6 Let $f(x, y) = x^2 + 3xy - y^2$. Compare the values of dz and Δz when x changes from 2 to 2.05 and y changes from 3 to 2.96.

3.4 Total Derivatives of vector functions and Jacobian Matrix: A linear approximation approach

First, we consider a real valued function of single variable. Assume that $f: \mathbb{R} \longrightarrow \mathbb{R}$ is a differentiable at a point $a \in \mathbb{R}$. Then,

$$f'(a) = \lim_{h \to 0} \left(\frac{f(a + h) - f(a)}{h} \right)$$

exists. This says, we can approximate $f(x)$ by

$$f(a + h) \approx f(a) + f'(a)\, h$$

when $h \to 0$. We can write this as

$$\in(h) = f(a + h) - f(a) - f'(a)\, h,$$

where $\in(h)$ is the error in the approximation and

$$\lim_{h \to 0} \left(\frac{\in(h)}{h} \right) = 0.$$

On the other hand,

$$\lim_{h \to 0} \left(\frac{f(a + h) - f(a) - f'(a)\, h}{h} \right) = 0.$$

Let us now generalize this for a vector valued function of several variables.

Definition 3.4 Let $\underline{f}: \mathbb{R}^n \to \mathbb{R}^m$ be a map and let $\underline{a}$ be a point in $\mathbb{R}^n$. We say that f is differentiable at $\underline{a}$, if there is a linear map $\underline{L}: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\underline{f}(\underline{a} + \underline{h}) \approx \underline{f}(\underline{a}) + \underline{L}(\underline{h}).$$

i.e. .

$$\lim_{\underline{h} \to \underline{0}} \left(\frac{\underline{f}(\underline{a} + \underline{h}) - \underline{f}(\underline{a}) - \underline{L}(\underline{h})}{\|h\|} \right) = \underline{0}.$$

In this case, we say that $\underline{L}$ is the total derivative of $\underline{f}$ at the point $\underline{a}$ and we write $D\underline{f}(\underline{a})$ to denote $\underline{L}$.

Remark: If $m = 1$ (single valued functions of several variables), the mapping $Df(\underline{a})$ is given by

$$Df(\underline{a})(h_1, h_2, \cdots, h_n) = \frac{\partial f}{\partial x_1}(\underline{a})h_1 + \frac{\partial f}{\partial x_2}(\underline{a})h_2 + \cdots + \frac{\partial f}{\partial x_n}(\underline{a})h_n.$$

Example 3.7 Consider the function $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (x + y^2, x^3 + 5y)$. Find total derivative of f at $(1,1)$.

Example 3.8 Suppose that $f: \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function, $f(1,1) = (5,8)$ and the jacobian matrix of f at $(1,1)$ is $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$. Estimate $f(1.1, 1.2)$.

Note: This is a good approximation only h_1=0.1, $h_2 = 0.2$ can be considered as very small numbers.

Theorem 3.3 Suppose that $f: \mathbb{R}^n \to \mathbb{R}^m$. Write $f = (f_1, f_2, \cdots, f_m)$, where each $f_i = \mathbb{R}^n \to \mathbb{R}$. If for all i and j, $\frac{\partial f_i}{\partial x_j}$ is defined and continuous lose to $\underline{a}$. Then, the function f is differentiable at $\underline{a}$, and the matrix for $Df(\underline{a})$ is given by

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}_{m\times n}.$$

This matrix is called the total derivative matrix or Jacobian matrix of f at $\underline{a}$.

Example 3.9 Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ defines the transformation from cylindrical polar coordinates to rectangular caartesian coordinates, Find $Df\left(5, \frac{\pi}{3}, 0\right)$.

3.5 Differentials of Higher order

Let $z = z(x,y)$ and it is differentiable at a point (x,y). Then, we have

$$dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy$$

If we treat dx, dy as constant and $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ as a function of x and y, then dz itself a function of x and y and it self-differentiable. Therefore, the second order differential

$$\begin{aligned} d^2z &= d(dz) \\ &= d\left[\frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy\right] \\ &= d\left[\frac{\partial z}{\partial x}\right] dx + d\left[\frac{\partial z}{\partial y}\right] dy \qquad \ldots\ldots..(1) \end{aligned}$$

Define the operator $d \equiv \frac{\partial}{\partial x}\, dx + \frac{\partial}{\partial y}\, dy$. Then,

$$d\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) dx + \frac{\partial z}{\partial y}\left(\frac{\partial z}{\partial x}\right) dy$$

$$= \frac{\partial^2 z}{\partial x^2} dx + \frac{\partial^2 z}{\partial y\,\partial x} dy \qquad \ldots\ldots..(2)$$

and

$$d\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) dx + \frac{\partial z}{\partial y}\left(\frac{\partial z}{\partial y}\right) dy$$

$$= \frac{\partial^2 z}{\partial x\,\partial y} dx + \frac{\partial^2 z}{\partial y^2} dy \qquad \ldots\ldots..(3)$$

Since $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ are differentiable, we have

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x} \ldots\ldots\ldots\ldots.(4)$$

By (1), (2), (3) & (4) we get

$$d^2z = \left[\frac{\partial^2 z}{\partial x^2} dx + \frac{\partial^2 z}{\partial y\,\partial x} dy\right] dx + \left[\frac{\partial^2 z}{\partial x\,\partial y} dx + \frac{\partial^2 z}{\partial y^2} dy\right] dy$$

$$= \frac{\partial^2 z}{\partial x^2} dx^2 + 2\frac{\partial^2 z}{\partial x\,\partial y} dx\,dy + \frac{\partial^2 z}{\partial y^2} dy^2$$

$$= \left[\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right]^2 z.$$

By the principle of Mathematical induction, we can prove, in general

$$d^n z = \left[\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right]^n z.$$

Remark:

If dx and dy are can't be treated as constant, then

$$d^2z = d(dz)$$

$$= d\left(\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy\right)$$

$$= d\left(\frac{\partial z}{\partial x}\right) dx + \frac{\partial z}{\partial x} d^2x + d\left(\frac{\partial z}{\partial y}\right) dy + \frac{\partial z}{\partial y} d^2y$$

$$= \left[\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right]^2 z + \frac{\partial z}{\partial x} d^2x + \frac{\partial z}{\partial y} d^2y$$

Higher order derivatives can be found in a similar manner and no simple general formula can be given to $d^n z$.

3.6 Derivatives of Composition Functions – Chain Rules

Theorem 3.4 Chain Rule 1

Suppose that $z = f(x, y)$ is differentiable function of the variables x and y, where $x = g(t)$, $y = h(t)$ are both differentiable functions of the variables t, then z is differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Example 3.10 Let $z = x^2y + 3xy^4$, where $x = \sin 2t$, $y = \cos t$. Find $\frac{dz}{dt}$.

Theorem 3.5 Chain Rule II

Suppose that $z = f(x, y)$ is differentiable function of the variables x and y, where $x = g(u, v)$, $y = h(u, v)$ are both differentiable functions of the variables u, v, then z is differentiable function of u, v and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u},$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

Example 3.11 If f is a differentiable function and $z = f(x^2 y)$, show that $x\frac{\partial z}{\partial x} = 2y\frac{\partial z}{\partial y}$.

Example 3.12 Let $F(x, y)$ be a homogeneous function of degree n. prove that

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = nF.$$

Hence show that for $F(x, y) = x^4 y^2 \sin^{-1}\left(\frac{y}{x}\right)$,

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = 6F.$$

3.7 Differentiation of Implicit Functions

Consider the function $f(x, y) = 0$, where $y = y(x)$. Differentiating with respect to x, we get

$$\frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0.$$

This gives,

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}.$$

Example 3.13 If $x^3 + y^3 = 6xy$, find $\frac{dy}{dx}$.

Now, let $f(x, y, z) = 0$, where $z = z(x, y)$. Then, differentiating with respect to x, we get

$$\frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0.$$

However, we know that $\frac{dx}{dx} = 1$ and $\frac{dy}{dx} = 0$. Thus,

$$\frac{\partial z}{\partial x} = - -\frac{f_x}{f_z}.$$

Similarly, we may obtain

$$\frac{\partial z}{\partial y} = - -\frac{f_y}{f_z}$$

3.8 Taylor's Theorem for Function of Two Variables

Theorem 3.6 Let (a, b) and its neighboring point $(a + h, b + k)$ be in the domain D of a function $f(x, y)$ which possess continuous partial derivatives of order n in D. Then, there is a number θ such that

$$f(a+h,b+k)=f(a,b)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a,b)+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a,b)+\cdots$$
$$\cdots+\frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n-1} f(a,b)+R_n,$$
where R_n is the remainder after n terms.

Cauchy's form for remainder is
$$R_n=\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+\theta h,b+\theta k), \qquad 0<\theta<1.$$

Proof: Let $x=a+th,\ y=b+tk,$ where $0\le t\le 1$. Then,
$$f(x,y)=f(a+th,b+tk)=\phi(t),$$
where $\phi(t)$ is a function of single variable.

Since partial derivatives of $f(x,y)$ are continuous up to order n, $\phi(t),\ \phi'(t),\cdots,\phi^n(t)$ are continuous on $[0,1]$. Now,
$$\phi'(t)=\frac{d\phi}{dt}=\frac{df}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$
$$=h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}$$
$$=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f.$$

We may take the differential operator
$$\frac{d}{dt}\equiv h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}.$$

Hence
$$\phi''(t)=\frac{d^2\phi}{dt^2}=\frac{d}{dt}\left(\frac{df}{dt}\right)$$
$$=\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)f$$
$$=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f.$$

Continuing in this way, we get
$$\phi^{(n)}(t)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+th,b+tk) \quad -----(*).$$

H.W. Prove this by the Principle of Mathematical Induction.

The Maclaurin series for single variable is
$$\phi(t)=\phi(0)+t\phi'(0)+\frac{t^2}{2!}\phi''(0)+\cdots+\frac{t^{n-1}}{(n-1)!}\phi^{(n-1)}(0)+\frac{t^n}{n!}\phi^{(n)}(\theta t),$$
where $0<\theta<1$.

Putting $t=1$ with $(*)$, we get

$$\phi(1) = f(a+h, b+k)$$
$$= f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) + \cdots$$
$$\cdots + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a,b) + R_n,$$

where

$$R_n = \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k), \qquad 0 < \theta < 1.$$

Remark: Putting $a+h=x,\ b+k=y$ or $h=x-a,\ k=y-b$, we get

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b)$$
$$+ \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a,b) + \cdots$$
$$+ \frac{1}{(n-1)!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^{n-1} f(a,b) + R_n,$$

where

$$R_n = \frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^n f(a+(x-a)\theta, b+(y-b)\theta), \qquad 0 < \theta < 1.$$

Example 3.14 Expand $x^2y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$.