SOUTH EASTERN UNIVERSITY OF SRI LANKA
Faculty of Applied Sciences

MTC 12011 MATHEMATICS FOR BIOLOGY II

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3 Determinant of a Matrix

The determinant of a square matrix is a unique number that associates with it. We shall use $\det(A)$ or $|A|$ to denote the determinant of a square matrix A of size n.

3.1 Determinant of a Matrix of order 1 and 2

Definition 3.1 The determinant of a scalar matrix (a) is the number a itself. The determinant of a 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is defined as

$$\det(A) = |A| = a_{11}a_{22} - a_{12}a_{21}.$$

Example 3.1 Compute the determinant of each of the following matrices:
(i) $A = \begin{pmatrix} 3 & 2 \\ -9 & 5 \end{pmatrix}$, (ii) $B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, (iii) $C = \begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix}$.

Remark 3.1: Note that determinant of a matrix can be a positive, zero or negative number.

3.2 Method of Cofactor Expansion

For $n \geq 3$, finding determinant is not direct forward. We first find "minors" and "cofactors" to find a higher dimensional matrices.

Definition 3.2 A *minor* is the determinant of the square matrix formed by deleting one row and one column from some larger square matrix. The minor corresponding to a_{ij}th entry of the matrix A of size $n \times n$ is the determinant of $(n-1) \times (n-1)$ matrix obtained by deleting i^{th} row and j^{th} column of A and is denoted by M_{ij}.

The *cofactor* C_{ij} of the entry a_{ij} is defined as $C_{ij} = (-1)^{i+j} M_{ij}$.

Definition 3.3 The determinant of an $n \times n$ matrix $A = (a_{ij})$ is defined as

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$$

and the procedure is known as the method of cofactor expansion (or Laplace expansion) about i^{th} row.

Remark 3.2: Similarly one can evaluate the determinant by expanding about a suitable column of the matrix. Corresponding equation for the expansion about j^{th} column is

$$\det(A) = \sum_{i=1}^{n} a_{ij}C_{ij}$$

Example 3.2 Compute the determinant of each of the following matrices by the cofactor expansion.

(i) $\begin{pmatrix} 3 & 5 & 4 \\ -2 & -1 & 8 \\ -11 & 1 & 7 \end{pmatrix}$, (ii) $\begin{pmatrix} 1 & 0 & 3 & 7 \\ 5 & 4 & 1 & 2 \\ 5 & 0 & 0 & 1 \\ 3 & 4 & 1 & 3 \end{pmatrix}$.

3.3 Properties of a Determinant

Let A be any $n \times n$ matrix.

1. Determinant of the identity matrix is 1.
2. $\det(A^T) = \det A$.
3. If each of the elements of any row is equal to 0, then $\det A = 0$.
4. If two rows of A are identical or one row is a multiple of another, then $\det A = 0$.
5. If A and B are two $n \times n$ square matrices, then
$$\det AB = \det A \cdot \det B$$
6. Let the matrix B be obtained by interchanging two rows of the matrix A, then
$$\det B = -\det A.$$
7. If A is a triangular matrix, then $\det A = a_{11}a_{22} \cdots a_{nn}$, the product of the triangular enries.

Example 3.3 If A is an 4×4 matrix with $\det(A) = 2$, then find $\det(3A^2A^T)$.

Example 3.4 Find the determinant of the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ -5 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}$.

Deduce the determinants of $B = \begin{pmatrix} 2 & -5 & 1 \\ 1 & 3 & 0 \\ 0 & 4 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 & 2 \\ -5 & 3 & 4 \\ 2 & 1 & 0 \end{pmatrix}$.

Example 3.5 Evaluate the determinant of $A = \begin{pmatrix} 2 & 0 & 0 \\ -5 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.

EXERCISES 2

1. Evaluate the following determinants.

$(a)\ \begin{vmatrix} 3 & 5 \\ -2 & -6 \end{vmatrix} \quad (b)\ \begin{vmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{vmatrix} \quad (c)\ \begin{vmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 5 \end{vmatrix} \quad (d)\ \begin{vmatrix} 2 & 0 & 1 & -1 \\ 0 & 2 & 0 & 0 \\ 3 & 1 & 2 & 0 \\ 1 & 4 & -2 & 3 \end{vmatrix}.$

2. Find the all values of x such that $\det(A - xI) = 0$, where $A = \begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix}$.

3. Evaluate the following determinants by inspection. Justify your answer.

$(a)\ \begin{vmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 5 & 5 \end{vmatrix} \quad (b)\ \begin{vmatrix} 0 & 5 & 0 & 1 \\ 4 & 4 & 0 & -4 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 5 \end{vmatrix} \quad (c)\ \begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 0 & 7 & 3 \end{vmatrix}.$

4. Let A and B be any two square matrices of same size. Which of the following are true? Which are false? If false, Give a counter example.
   (a) $\det(A + B) = \det(A) + \det(B)$.
   (b) $\det(AB) = \det(A)\ \det(B)$.
   (c) $\det(AB) = \det(BA)$.

5. Evaluate $\begin{vmatrix} 3 & 1 & 2 \\ 2 & 4 & 5 \\ 1 & 1 & 3 \end{vmatrix}$. Deduce $\begin{vmatrix} 2 & 4 & 5 \\ 1 & 1 & 3 \\ 3 & 1 & 2 \end{vmatrix}$ and $\begin{vmatrix} 3 & 2 & 1 \\ 1 & 4 & 1 \\ 2 & 5 & 3 \end{vmatrix}$.

6. Let $A = \begin{pmatrix} 1 & 0 & 50 \\ 0 & 7 + x & -3 \\ 0 & 4 & x \end{pmatrix}$. Find all values of x such that $\det(A) = 0$.