4 EXTREMA

4.1 Local Extremum

Definition 4.1 A function $f(x,y)$ is said to have a local maximum (or minimum) at the point (a,b) if $f(x,y) \leq f(a,b)$ $\big(f(x,y) \geq f(a,b)\big)$ for all points (x,y) in some disk with center (a,b). The number $f(a,b)$ is called a local maximum (local minimum) value.

Figure

Theorem 4.1 If the function $f(x,y)$ has a local maximum or local minimum at the point (a,b) and $f_x(a,b)$, $f_y(a,b)$ exist, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$. i.e. $\nabla f(a,b) = 0$.

Definition 4.2 A point (a,b) is called a critical point of the function $f(x,y)$ if $f_x(a,b) = 0$ and $f_y(a,b) = 0$ or if at least one of these partial derivatives does not exist.

Remark 4.1 All extremum points of $f(x,y)$, if exist, are critical points. However, a critical point of f could have a local maximum or local minimum or neither.

Example 4.1 Let $f(x,y) = \begin{cases} 1 & ; \quad xy \neq 0 \\ 0 & ; \quad xy = 0 \end{cases}$. Show that the origin is a critical point of f, but it has no extremum there at.

Example 4.2 Let $f(x,y) = |x| + |y|$ for all $(x,y) \in \mathbb{R}^2$. Show that the origin is a critical point of f and f has a local minima there at.

4.2 Sufficient Conditions for Extrema

Let $f_x(a,b) = 0 = f_y(a,b)$ $-----$ (1)

Assume that $f(x,y)$ have continuous partial derivatives in a neighborhood of $f(a,b)$ and f_{xx}, f_{xy}, f_{yy} are not zero simultaneously at (a,b). Then, by Taylor's Theorem, we have $0 < \theta < 1$ such that

$$f(a+h,b+k) = f(a,b) + \left(h\frac{\partial}{\partial x}f(a,b) + k\frac{\partial}{\partial y}f(a,b)\right)$$
$$+\frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2}f(a+\theta h,b+\theta k) + 2hk\frac{\partial^2 f}{\partial xy}f(a+\theta h,b+\theta k) + k^2\frac{\partial^2 f}{\partial y^2}f(a+\theta h,b+\theta k)\right)$$
$$----- \quad (2)$$

Since f has continuous second order partial derivatives, we have

$f_{xx}(a+\theta h,b+\theta k) \longrightarrow f_{xx}(a,b)$,

$f_{xy}(a+\theta h,b+\theta k) \longrightarrow f_{xy}(a,b)$ and

$f_{yy}(a+\theta h,b+\theta k) \longrightarrow f_{yy}(a,b)$ as $(h,k) \longrightarrow (0,0)$ $-----$ (3).

By (1), (2) and (3), we get

$$\Delta = f(a+h, b+k) - f(a,b) = \frac{1}{2}(h^2A + 2hkB + k^2C),$$
where $A = f_{xx}(a,b)$, $B = f_{xy}(a,b)$ and $C = f_{yy}(a,b)$.

We first assume that $A \neq 0$. Then,
$$\Delta = \frac{1}{2A}(h^2A^2 + 2hkAB + k^2AC) = \frac{1}{2A}[(Ah+Bk)^2 + k^2(AC - B^2)].$$

Case I: $AC - B^2 > 0$, $A \neq 0$ for all h, k.
- (a) If $A > 0$, then $\Delta > 0$. Hence $f(a,b)$ is minimum.
- (b) If $A < 0$, then $\Delta < 0$. Hence $f(a,b)$ is maximum.

Case II: $AC - B^2 < 0$, $A \neq 0$ for all h, k.
In this case, the sign of Δ depends on h, k. Therefore, $f(a,b)$ is not an extremum value.

Case III: $AC - B^2 = 0$, , $A \neq 0$.
In this case, Δ can be zero for some h and k. Therefore, further investigation is required to determine the extrema.

If $A = 0$, then
$$\Delta = \frac{1}{2}(2Bhk + k^2C) = Bhk + \frac{k^2}{2}C.$$
Note that $(h,k) \neq (0,0)$.

Case IV: $A = 0$, $B \neq 0$
In this case, the sign of Δ depends on h, k. Therefore, $f(a,b)$ is not an extremum value.

Case V: $A = 0, B = 0$
In this case, $\Delta = \frac{k^2}{2}C$ can be zero for some h and k. Therefore, further investigation is required to determine the extrema.

Summary:

A	B	$AC - B^2$	**Remark**
> 0	$-$	> 0	$f(a,b)$ is minimum
< 0	$-$	> 0	$f(a,b)$ is maximum
$\neq 0$	$-$	< 0	Not extremum. A saddle point
$\neq 0$	$-$	$= 0$	Need further investigation
$= 0$	$\neq 0$	$-$	Not extremum. A saddle point
$= 0$	$= 0$	$-$	Need further investigation

Example 4.3 Find and classify the critical points of $f(x,y) = x^2 + y^2 - 2x - 6y + 14$.

Example 4.4 Find and classify the critical points of $f(x,y) = x^3 + y^3 - 3x - 12y + 20$.

Example 4.5 In a triangle XYZ, find the maximum value of $\cos x \cos y \cos z$.

Example 4.6 Find and classify the critical points of $f(x,y) = y^2 + x^2 y + x^4$.

4.3 Lagrange's Multipliers Method

The Lagrange's multipliers method is used to find the extrema of a function of several variables subject to some given constraints.

Solving Procedure:

Suppose that we need to optimize (maximize or minimize) a function $f(\underline{x})$; $\underline{x} = (x_1, x_2, \cdots, x_m)$ subject to m constraints $\phi_i(\underline{x}) = 0; i = 1, 2, \cdots, m$.

Step 1: Set

$$F(\underline{x}) = f(\underline{x}) + \sum_{j=1}^{m} \lambda_j \phi_j.$$

Step 2: Differentiate F with respect to each variable x_i and equate to zero.

$$\frac{\partial F}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial \phi_j}{\partial x_i} = 0.$$

Step 3: Solve all equations in step 2 along with the constraints $\phi_i(\underline{x}) = 0$ for each $x_i, \lambda_j, i, j = 1, 2, \cdots, m$.

A stationary point of $f(\underline{x})$ will be extrema if

$$d^2 F(\underline{x}) = \left(\sum_{j=1}^{m} \frac{\partial}{\partial x_i} dx_i \right)^2 F(\underline{x})$$

keeps the same sign. If $d^2 F > 0$, then it will be minima and if $d^2 F < 0$, then it will be maxima.

Example 4.7 A rectangular box opened at the top is to have the volume of 32 cubic units. Find the dimensions of the box requiring least material for its construction.

Example 4.8 Find the volume of the greatest rectangular parallelepiped that can be inscribed in the ellipsoid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

4.4 Sufficient condition for a general problem

A sufficient condition for $F(\underline{x})$ to have local minima(maxima)at $\underline{x}^*$ is each root of the polynomial in ε in the determinant should be positive (negative).

$$\begin{vmatrix} F_{xx}-\varepsilon & F_{xy} & F_{xz} & \phi_{1x} & \phi_{2x} & \phi_{3x} \\ F_{yx} & F_{yy}-\varepsilon & F_{yz} & \phi_{1y} & \phi_{2y} & \phi_{3y} \\ F_{zx} & F_{zy} & F_{zz}-\varepsilon & \phi_{1z} & \phi_{2z} & \phi_{3z} \\ \phi_{1x} & \phi_{1y} & \phi_{1z} & 0 & 0 & 0 \\ \phi_{2x} & \phi_{2y} & \phi_{2z} & 0 & 0 & 0 \\ \phi_{3x} & \phi_{3y} & \phi_{z} & 0 & 0 & 0 \end{vmatrix} = 0$$

Example 4.9 Optimize the function $f(x) = -3x^2 - 6xy - 5y^2 + 7x + 5y$ subject to the condition $x + y = 5$.