5 IMPLICIT AND INVERSE FUNCTION THEOREMS

5.1 Introduction to Implicit Function Theorems

An equation of the form $F(x, y)$ does not necessarily represents a unique function. For example, let us consider the equation $F(x, y) = x^2 + y^2 - 1 = 0,\ x \in [-1, 1]$. Solving for y, we obtain

$$y = \pm\sqrt{1 - x^2}.$$

That is, for one value of x, we get two values of y. So that is not a function.

A natural question arises in the study if implicit function is under which condition(s) the implicit function $F(x, y) = 0$ can be locally expressed explicitly in the form $y = f(x)$. This leads to the study of implicit function theorem.

5.2 Implicit function theorem for two variables

Let F be a real valued continuous function defined on some neighborhood N of the points(a, b). If

(i) $F(a, b) = 0$,

(ii) $\frac{\partial F}{\partial y}$ exists and is continuous on N, and

(iii) $\frac{\partial F}{\partial y}(a, b) \neq 0$,

then there exists a unique function g defined on some neighborhood N_a of a such that

(i) $g(a) = b$

(ii) $F(x, g(x)) = 0$ for each $x \in N_a$, and

(iii) g is continuous.

More over if $\frac{\partial F}{\partial x}$ also exists and is continuous on N then, g is continuously differentiable on N_a and g' is giving by

$$g'(t) = \frac{\frac{\partial}{\partial x} F(t, g(t))}{\frac{\partial}{\partial y} F(t, g(t))};\ \ t \in N_a$$

Example 5.1 Verify the implicit function theorem for the equation $F(x, y) = xy + x^2 = 0$.

Example 5.2 Show that there exists a continuously differentiable function g defined by the equation $F(x, y) = x^3 + y^3 - 3xy - 4 = 0$ in a neighborhood $x = 2$ such that $g(2) = 2$ and also find its derivative.

5.3 Implicit function theorem for 3 variables

Let $F(x, y, z)$ be a real valued continuous function of three variables defined on some neighborhood N of a point $P_0(x_0, y_0, z_0)$ in $\mathbb{R}^3$. If

(i) $F(P_0) = 0$,

(ii) F is continuously differentiable on N, and

(iii) $\frac{\partial F}{\partial z}(P_0) \neq 0$ or the Jacobian

$$\begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \end{vmatrix} = \frac{\partial F(x,y,z)}{\partial z} \neq 0$$

then, there exists a unique function g in a nbhd N_0 of (x_0, y_0) in $\mathbb{R}^2$ such that

(i) $g(x_0, y_0) = z_0$ and

(ii) $F(x, y, g(x, y)) = 0$ for $(x, y) \in N_0$.

Example 5.3 Consider a function $f: \mathbb{R}^3 \to \mathbb{R}$, $F(x, y, z) = x^2 + y^2 + z^2 - 1$. Decide whether the equation $F(x, y, z) = 0$ defines a unique function g in a nbhd of $\left(\frac{1}{2}, \frac{1}{2}\right)$ such that $\left(\frac{1}{2}, \frac{1}{2}\right) = -\frac{1}{\sqrt{2}}$.

Example 5.4 Let $f(x, y, z)$ be a continuously differentiable function of one variable such that $f(1) = 0$. Find the conditions $F(x, y, z) = f(xy) + f(yz)$ can be solved for z in a neighborhood of $(1,1,1)$.

5.4 Implicit Functions theorem

Let $F_1(\underline{x}, \underline{u}), F_2(\underline{x}, \underline{u}), \dots, F_m(\underline{x}, \underline{u})$ be m functions of $n + m$ variables, where $\underline{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $\underline{u} = (u_1, u_2, \dots, u_m) \in \mathbb{R}^m$ defined in a neighborhood N of the point $(\underline{a}, \underline{u})$ with $\underline{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ such that

(i) $F(\underline{a}, \underline{u}) = 0$,

(ii) F_j is continuously differentiable for each ; $1 \leq j \leq m$,

(iii) The Jacobian $\frac{\partial(F_1, F_2, \dots, F_m)}{\partial(u_1, u_2, \dots, u_m)} \neq 0$, at the point $(\underline{a}, \underline{u}) \in \mathbb{R}^{n+m}$.

Then, there exist exactly m functions g_i of n variables such that each g_i is defined in the neighborhood S of $\underline{a}$ and

(i) $g_i(a_1, a_2, \dots, a_n) = (u_1, u_2, \dots, u_m)$; $1 \leq i \leq m$,.

(ii) $F(x_1, x_2, \dots, x_n, g_1, g_2, \dots, g_m) = 0$ for each $(x_1, x_2, \dots, x_n) \in S$, and

(iii) Each g_i is continuously differentiable.

Theorem: Suppose that $u = f(x, y)$ and $v = g(x, y)$ are real valued continuously differentiable function defined in some open sphere S. If the Jacobian $\frac{\partial(u,v)}{\partial(x,y)} = 0$ for each $(x, y) \in S$, then u and v are functionally dependent.

Example 5.5 Show that there is a functional relationship between $u = 2\ln x + \ln y$, $v = e^{x\sqrt{y}}$; $x, y > 0$ and determine it.

Remark:

(i) The hypothesis $\frac{\partial(u,v)}{\partial(x,y)} = 0$ on an open sphere in the above theorem is necessary as well.

Illustrative Example

Suppose that $u = x^3, v = y^3$. It is clear that u and v are functionally independent. So,

We need Jacobian not equal zero.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 3x^2 & 0 \\ 0 & 3y^2 \end{vmatrix} = 9x^2y^2 \neq 0 \text{ except the origin.}$$

(ii) Theorem can be extended for function of three variable.

5.5 Introduction to Inverse Function Theorem

Recall: For functions of single variable, suppose $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}$ open subset, is continuously differentiable function. Then,

- If $f'(x_0) \neq 0$ for any point $x_0 \in D$, then $f'(x_0) \neq 0$ in $(x_0 - \delta, x_0 + \delta) = I \subseteq D$ for some $\delta > 0$.
- In fact $f'(x)$ has the same sign of $f'(x_0)$ in I.
- If $f'(x) > 0$ then $f(x)$ is strictly increasing in I.
- If $f'(x) < 0$ then $f(x)$ is strictly decreasing in I.
- In both cases f is one to one on I.

Remark:

Since I is open containing $x_0, f(I)$ is open containing $f(x_0)$.
Therefore the function from I to $f(I)$ is 1-1 and onto and hence is invertible on I.
Moreover, the inverse $f^{-1}(x)$ is differentiable on $f(I)$ at $f(x_0)$.
Thus, If $f'(x_0) \neq 0$ for every $x_0 \in D$ then the above results hold for every points of I.

Theorem If $f^{-1}: f(I) \longrightarrow I$ is continuously differentiable at $f(x_0)$, then

$$(f^{-1})'f(x_0) = \frac{1}{f'(x_0)}.$$

Example 5.6 Verify the above theorem for $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \sin x$ and $x_0 = \pi/6$.

A similar result is true for functions of several variables and is known as inverse function theorem.

Definition 5.1 (Inverse)

A function f with domain $D \subseteq \mathbb{R}^n$ and range $D^* \subseteq \mathbb{R}^n$ is said to be invertible on D if there exist a function $g: D^* \to D$ such that

$g(f(P)) = P,$ and

$f(g(Q)) = Q$

for every $P \in D, Q \in D^*$

Recall that $f: D \to D^*$ is invertible on D if and only if f is 1-1 (it is already onto). Moreover, the function g is uniquely determined by f and is called inverse of f and denoted by $g(x) = f^{-1}(x)$.

Example 5.7 Let $D \subseteq \mathbb{R}^2$ consisting of all pairs (r, θ) with $r > 0$ and $0 < \theta < \pi$. Define a function $f(r, \theta)$ on D such that $f: D \to D^*$ by

$$f(r,\theta) = \big(x(r,\theta), y(r,\theta)\big) = (r\cos\theta, r\sin\theta).$$

Discuss the invertibility of f.

Remark:

So far by a neighborhood of a point $\underline{a} \in \mathbb{R}^n$, we meant an open sphere $S\big(\underline{a}, r\big) = \{x \,/\, \|\underline{x} - \underline{a}\| < r\}$ centered at $\underline{a}$ and radius r. From now onwards, if necessary, we shall regard those sets u of $\mathbb{R}^n$ as a neighborhood of $\underline{a}$ which contain an open sphere $S(\underline{a}, \underline{r})$ for a suitable r. For example, both $\{(x,y)/\sqrt{(x-2)^2 + y^2} < r = S(\underline{a}, \underline{r})$ and $\{(x,y)/\sqrt{(x-2)^2 + y^2} \le r$ are neighborhoods of a point $\underline{a} = (2,0)$ on $\mathbb{R}^2$.

Definition 5.2 (Invertible)

Let $: D \to \mathbb{R}^n$; $D \subset \mathbb{R}^n$, we say that f is locally invertible at a point $P \in D$ if there exist a neighborhood N of P contained in D and neighborhood N^* of $f(P)$ such that

(i) $f(N) = N^*$,

(ii) f is one to one on N.

Example 5.8 Consider the map $f(x,y) = \{2xy, x^2 - y^2\}$ defined from $\mathbb{R}^2$ to $\mathbb{R}^2$. Show that f is not invertible. Define $D = \{(x,y) | x > 0\}$. Show that f is invertible on D and find its inverse.

5.6 Inverse function theorem

Let $F_1, F_2, \dots, F_n$ be n real valued functions on an open subset of $\mathbb{R}^n$. Let $\underline{F} = (F_1, F_2, \dots, F_n)$ be a function from $\mathbb{R}^n$ to $\mathbb{R}^n$ with domain D. If F is continuously differentiable at a point $P_0 = (a_1, a_2, \dots, a_n) \in D$ and if the Jacobian $\frac{\partial(F_1, F_2, \dots, F_n)}{\partial(x_1, x_2, \dots, x_n)} \neq 0$ at P_0 then the function F is locally invertible at P_0. Moreover the local inverse F^{-1} of F is continuously differentiable at $F(P_0)$.

Example 5.9 Show that the function $f: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = (y\sin x, x + y + 1)$ is locally invertible at the point $(0,1)$.