6 INTEGRATION ON $\mathbb{R}^2$

6.1 Some Basic Definitions

1. Domain : A domain is an open connected set of points.

Any two points of a set can be joint by a broken line with finite number of segments all of whose points belong to the set.

2. Boundary points: A point is called a boundary point of a set S if every neighborhood of it contains points of S as well as those not belongs to S. The set of all boundary points of S is called the boundary of S.

3. Region : Closed domain (include the boundary points also). That is,

$$\text{Region} = \text{Domain} \cup \text{Boundary}.$$

Simple connected / Multiple conncted region: A region is simply connected if the boundary consists of a single simply connected curve; i.e. any two points of the set can be connected by one straight line segment. Otherwise, the region is called a multiple connected region..

4. Regular (Quadratic) region: A region E is said be regular (quadratic) with respect to $x-$ axis if it is bounded by the curves $y = \phi(x)$ and $y = \psi(x)$, $x = a$ and $x = b$, where ϕ, ψ are continuous functions such that $\phi(x) \leq \psi(x) \quad \forall\, x \in [a, b]$.

Remark In this case, a line drawn parallel to $y-$ axis between $x = a$ and $x = b$ meets the boundary of E exactly in 2 points .

Regular region with respect to $x-$axis is similarly defined. i.e. the region is bounded by the curves $x = f(y), x = g(y)$, $y = c$ and $y = d$ such that $f(y) \leq g(y)\ \forall y \in [c, d]$.

A region is called regular if it is regular with respect to both axes.

5. Piece –wise Regular : If the region can be divided into finite numbers of regular regions with respect to a particular axis then it is called piece-wise regular with respect to that axis.

6. Positive Sense: The contour of a region is said to described in the positive sense if the end closed region always lies to the left as one integrating along the curve.

Remark The positive sense of the boundary of a simple connected region is anti-clock wise direction. Area S enclosed the region is then taken to be positive.

In a multiple connected region,

7. Diameter of a region E: Diameter of a region E is the length of the largest line segment that joins two points of E.

Observe that the diameter tends to zero implies that the area tends to zero. However, the area tends to zero does not imply that the diameter tends to zero

6.2 Evaluating Double Integral of a Region

Theorem 6.1 If a double integral over the region E,

$$\iint_E f(x,y)\,dx\,dy$$

exists for a function $f(x,y)$ defined on the regular region E bounded by the curves $y=\phi(x), y=\psi(x), x=a, x=b$ where ϕ,ψ are continuous functions such that $\phi(x)\le\psi(x)\quad \forall\, x\in[a,b]$ and if the integral $\int_{\emptyset(x)}^{\varphi(x)} f\,dy$ exists then, $\int_a^b \left\{\int_{\emptyset(x)}^{\varphi(x)} f\,dy\right\} dx$ exist and

$$\iint_E f(x,y)\,dx\,dy = \int_a^b \left\{\int_{\emptyset(x)}^{\varphi(x)} f\,dy\right\} dx \;-----\; (1).$$

If instead, the region is bounded by $x=\phi_1(y)\;\; x=\psi_1(y), y=c\,, y=d\,;\; \phi_1,\psi_1$ are continuous $\forall\, y\in[c,d]$ such that $\phi_1(y)\le\psi_1(y)$ and if $\iint_E f(x,y)\,dx\,dy$ and $\int_{\emptyset_1(y)}^{\varphi_1(y)} f\,dx$ exist then,

$$\iint_E f(x,y)\,dx\,dy \;=\int_c^d \left\{\int_{\emptyset_1(y)}^{\varphi_1(y)} f\,dx\right\} dy \;-----\; (2).$$

Remarks

I. If E is regular with respect to y axis, then use equation (1).
II. If E is regular with respect to x axis, then use equation (2).
III. If E is not regular with respect to both axes and if we can divide divide the region E as the finite number of regular regions $E_1, E_2 \ldots E_n$ then,

$$\iint_{\boldsymbol{E}} = \iint_{\boldsymbol{E_1}} + \iint_{\boldsymbol{E_2}} + \ldots\ldots + \iint_{\boldsymbol{E_n}}.$$

Example 6.1 Evaluate double integral $\iint y\,dx\,dy$ over the part of the plane bounded by the line $y=x$ and the parabola $y=4x-x^2$.

Example 6.2 Find the value of $\iint_E e^{\frac{y}{x}}\,dx\,dy$ if the domin E of the integration is bounded by the straight line $y=x, y=0$ and $x=1$.

Example 6.3 Change the order of the integration of $\int_0^1 \left\{\int_x^{\sqrt{x}} f(x,y)dx\right\} dy$.

Example 6.4 Evaluate $\iint_E e^{x+y}\,dxdy,$ where E is the region lies between two squares of lengths 2 and 4 with centered at origin and sides are parallel to x and y axes.

Example 6.5 Evaluate $\iint_R f(x,y)dx\,dy; R\,\{[0,1]\times[0,1]\}$, where

$$f(x,y)=\begin{cases} x+y; & x^2<y<2x^2 \\ 0; & \text{otherwise}\end{cases}.$$

Example 6.6 Evaluate $\iint_R [x+y]dx\,dy\,; R=\{(x,y)|x\in[0,1], y\in[0,2]\}$; where $[x+y]$ is the greatest integer $\le x+y$.

Example 6.7 Prove that $\iint_R \sqrt{|y - x^2|}\ dy\,dx = \frac{3\pi+8}{8}$, where $R = \{(x,y)|x \in [-1,1], y \in [0,2]\}$.

6.3 Change of variables

Suppose that we need to change Cartesian coordinate system (x,y) to curvilinear coordinate system (u,v). Let coordinate transformation equations be

$$x = x(u,v), \qquad y = y(u,v).$$

Let E be the region bounded by the closed curve C in (x,y) plane and $1-1$ correspondence in (u,v) plane be the region R bounded by the closed curve C_1.

Let $x,\ y$ are continuous functions and have continuous first order partial derivatives at all points.

If $\iint_E f(x,y)\,dxdy$ is integrable in the region E and $f(x,y) = F(u,v)$, then

$$\iint_E f(x,y)\,dx\,dy = \iint_R F(u,v)|J|\,du\,dv,$$

where

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Example 6.8 Evaluate $\iint_E (x^2+y^2)dx\,dy$; $E: x^2+y^2 \le a^2$.

Example 6.9 Evaluate $\iint_{\boldsymbol{E}} \sqrt{\frac{a^2\,b^2 - b^2x^2 - a^2y^2}{a^2\,b^2 + b^2x^2 + a^2y^2}}\ dx\,dy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Example 6.10 **Evaluate** $\iint_E [2a^2 - 2a(x+y) - (x^2+y^2)]dx\,dy$, where $E: x^2+y^2+2a(x+y) \le a^2$.

Example 6.11 Evaluate $\iint_E (y-x)dx\,dy$, where E is the region bounded by line segments $y = x-3, y = x-1,\ 3x+y = 5$ and $3x+y = 7$.

Example 6.12 Evaluate $\iint_E x^3y^3\,dx\,dy$, where E is the region bounded by the parabolas $y^2 = ax,\ y^2 = bx;\ 0 < a < b,\ x^2 = py$ and $x^2 = qy;\ 0 < p < q$.

Example 6.13 Evaluate $\iint_E x^{m-1}y^{n-1}(1-x-y)^{p-1}\,dx\,dy,\ m \ge 1, n \ge 1, p \ge 1$, where E is the region bounded by $x = 0, y = 0, x+y = 1$

Example 6.14 Evaluate $\int_0^\pi \int_0^\pi cos|x+y|\,dx\,dy$.

6.4 Area of the plane Region

The area of the plane region is given by

$$A = \iint dx\,dy = \iint\limits_R |J|du\,dv.$$

Example 6.15 Find the area of the circle with radius a.