7 INTEGRATION ON $\mathbb{R}^3$

7.1 Line Integrals

Definition 7.1 Parametric Curves

A parametric curve in space is defined to be a vector valued function Γ whose domain is subset of $\mathbb{R}$ and the range is subset of $\mathbb{R}^3$. The curve is continuous if the function Γ is continuous and is called a Jordan arc if Γ is one to one.

Suppose that a curve Γ whose position vector $\underline{r} = (x, y, z)$ at any point can be parametrically represented by $\Gamma(\mathrm{t})$: $\underline{r} = \big(x(t), y(t), z(t)\big)$; $t \in [a, b]$, where t is a parameter. Throughout this chapter, we assume $x(t), y(t), z(t)$ are single valued continuous functions and are continuously differentiable (smooth) unless otherwise stated alternatively.

Theorem 7.1 Length of a curve
If $r(t)$ is a smooth curve in $\mathbb{R}^3$ such that $r'(t)$ exist and continuous then the length of the curve from the point $t = a$ to $t = b$ is given by

$$l(a, b) = \int_a^b |r'(t)| dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}.$$

Example 7.1 Find the length of the curve $x = at^2, y = 2at, z = at;\ 0 \le t \le 1$.

Definition 7.2 Line Integrals
Let C be a space curve defined by $x = x(t),\ y = y(t),\ z = z(t),\ a \le t \le b$. Let $F = (f, g, h)$ be a bounded vector valued function defined at every point of the curve C, where $f = f(x, y, z),\ g = g(x, y, z),\ h = h(x, y, z)$. Then, $\int_C (f dx + g dy + h dz)$ is called the line integral of $F = (f, g, h)$ along C.

REMARK To evaluate the line integral $L = \int_C (f dx + g dy + h dz)$:
If $\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$. Then,

$$\frac{d\underline{r}}{dt} = \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} + \frac{dz}{dt}\underline{k}.$$

Suppose that $\underline{F} = f\underline{i} + g\underline{j} + h\underline{k}$. Then,

$$\begin{aligned} L &= \int_C \left(f\underline{i} + g\underline{j} + h\underline{k}\right) \cdot \left(\frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} + \frac{dz}{dt}\underline{k}\right) dt \\ &= \int_c \left(\underline{F}.\frac{d\underline{r}}{dt}\right) dt \\ &= \int_c \underline{F}\,.\,d\underline{r} \end{aligned}$$

Example 7.2 Prove that $\int \frac{x^2+y^2}{p} ds = \frac{\pi ab}{4}\left[4+(a^2+b^2)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\right]$ when the integral is taken round the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose arc-length is denoted by s and p is the length of the perpendicular from the origin to the tangent of the ellipse.

Example 7.3 Show that $\int_c (y^2+z^2)dx+(z^2+x^2)dy+(x^2+y^2)dz=-2\pi ab^2$, where the curve c is the part for $z \geq 0$ at the intersection of the surfaces $x^2+y^2+z^2=2ax, x^2+y^2=2bx; a > b > 0$.

7.2 The Surface Area

Theorem 7.2 Let it be required to compute the area $\mathcal{S}$ of a surface bounded by a curve c. The surface being defined by the equation $z=\psi(x,y)$, where ψ is continuous and has continuous partial derivatives. Let Γ be the projection of c on the Oxy plane. Let D be the domain on the Oxy plane bounded by Γ and σ be the area of D. Then,

$$\mathcal{S}=\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dxdy.$$

Similarly, if the equation of the surface is of the form $x=\theta(y,z), y=\phi(x,z)$, then the corresponding formulae for calculating the surface area ares of the form

$$\mathcal{S}=\iint_D \sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\, dydz,$$

$$\mathcal{S}=\iint_D \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\, dxdz,$$

where D_1, D_2 are the domains in the $yz-$plane and $xz-$plane respectively in which the given surface is projected.

Example 7.4 Compute the surface area of the sphere $x^2+y^2+z^2=a^2$.

Example 7.5 The x *and* y coordinates of a point on the paraboloid $2z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ are expressed in the form $x=a\tan\theta\cos\phi, y=b\tan\theta\sin\phi$, where θ is the angle of inclination of the normal at any point on the z axis. Show that the area of the cap of the surface cut off by the curve $\theta=\lambda$ is $\frac{2\pi ab}{3}(\sec^3\lambda-1)$.

Example 7.6 Find the area of the surface of the cylinder $x^2+y^2=a^2$ which is cut off by the cylinder $x^2+z^2=a^2$.

7.3 Surface Integrals

7.3.1 Surface Integral of scalar functions

Definition 7.3 Let $\mathcal{S}$ be a (piece wise) smooth surface bounded by a (piece wise) smooth curve C. Let $f(x,y,z)$ be a bounded function defined at each point of the surface $\mathcal{S}$. Then the surface integral of the first type of the function f over the surface $\mathcal{S}$ is defined and denoted by

$$\iint_S f(x,y,z)d\,\mathcal{S} = \iint_D f\big(x,y,z(x,y)\big)\frac{dxdy}{cos\gamma},$$

where γ is the angle of inclination to the surface $\mathcal{S}$ with $z-$ axis and D is the projection of $\mathcal{S}$ on Oxy plane.

To evaluate the surface integral,

$$\iint_D f(x,y,z(x,y))\frac{dxdy}{cos\gamma} = \iint_D f(x,y,z(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy.$$

Remark:

If the surface is represented by $x = x(y,z)$ or $y = y(y,z)$, then

$$\iint_S f(x,y,z)d\,\mathcal{S} = \iint_{D_1} f(x(y,z),y,z)\sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\,dydz,$$

where D_1 is the projection of $\mathcal{S}$ on yz $-$plane, or

$$\iint_S f(x,y,z)d\,\mathcal{S} = \iint_{D_2} f(x,y(x,z),z)\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dxdz,$$

where D_2 is the projection of $\mathcal{S}$ on xz $-$plane.

Example 7.7 Evaluate the surface integral $\int_S \frac{1}{r} d\,\mathcal{S}$, where $\mathcal{S}$ is portion of the hyperbolic paraboloid $z = xy$ cut off by the cylinder $x^2+y^2=a^2$ and r is the distance from a point on the surface to the z $-$axis.

7.3.2 Surface Integrals of Vector Functions

For the vector functions, suppose $\underline{F}(x,y,z) = f(x,y,z)\,\underline{i} + g(x,y,z)\,\underline{j} + h(x,y,z)\,\underline{k}$ and $\underline{n} = \cos\alpha\underline{i} + \cos\beta\,\underline{j} + \cos\gamma\,\underline{k}$. We have $d\underline{s} = dydz\underline{i} + dxdz\underline{j} + dxdy\underline{k}$. Therefore,

$\underline{F}.\underline{n} = f\cos\alpha + g\cos\beta + h\cos\gamma$, and

$\underline{F}.d\underline{s} = fdydz + gdxdz + hdxdy.$

Then,

$$\int_S \underline{F}.\underline{n}d\underline{s} = \int_S \underline{F}d\underline{s}\,.$$

i.e.

$$\int_S (f\cos\alpha + g\cos\beta + h\cos\gamma)ds$$

$$= \int_S fdydz + gdxdz + hdxdy$$

$$= \int_{D_1} f[x(y,z),y,z])dydz + \int_{D_2} g[x,y(x,z),z])dxdz + \int_{D_3} h[x,y,z(x,y)])dxdy.$$

Example 7.8 Evaluate $\iint_s xdydz + dzdx + xz^2\,dxdy$, where s is the outer side of the part of the sphere $x^2+y^2+z^2=1$ in the first octant.

7.3.3 Surface Integral of Parametric Surfaces

If the surface is given parametrically by $x = x(u,v), y = y(u,v), z = z(u,v)$, $u, v \in D$, then

$$\iint_S h dxdy = \iint_{D^*} h[x, y, z(x,y)] \frac{\partial(x,y)}{\partial(u,v)} dudv - - - - - (1)$$

$$\iint_S f dydz = \iint_{D^*} f[x(y,z), y, z] \frac{\partial(y,z)}{\partial(u,v)} dudv - - - - - (2)$$

$$\iint_S g dxdz = \iint_{D^*} g[x, y(x,z), z] \frac{\partial(z,x)}{\partial(u,v)} dudv - - - - - (3)$$

$$\iint_S h dxdy + f dydz + g dxdz = (1) + (2) + (3)$$

where D^* is the region in uv −plane oriented in the same sense as $\mathcal{S}$.

Example 7.9 Evaluate $\int_S yzdydz + zxdzdx + xydxdy$, where $\mathcal{S}$ is the surface of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

7.4 Volumes by Double Integrals

7.4.1 Volume of the cylindrical solid

Definition 7.4 Let the volume be bounded above by the surface S: $z = \psi(x,y)$ and below by the projection D_1 of S on Oxy plane. Then, the volume V bounded by theses surfaces is given by

$$\begin{aligned} V &= \iint_{D_1} z dxdy \\ &= \iint_{D_1} \psi(x,y) dxdy \\ &= \iint_{D_1} z \cos\gamma \, ds. \end{aligned}$$

To evaluate this,

$$V = \int_{x=a}^{b} \left\{ \int_{y=\phi_1(x)}^{\phi_2(x)} \psi \, dy \right\} dx.$$

Exercises:

Write down the equation of a volume if it is

(i) bounded above by $x = \theta(y,z)$ and below by D_2.

(ii) bounded above by $y = \phi(x,z)$ and below byD_3.

7.4.2 Volume Bounded by two surfaces

Let the volume be bounded above by the surface S_1: $z = \psi_1(x,y)$ and below by S_2: $z = \psi_2(x,y)$. Then, the volume V bounded by theses surfaces is given by

$$\begin{aligned} V &= \iint_{D_1} z dxdy \\ &= \int_{x=a}^{b} \left\{ \int_{y=\phi_1(x)}^{\phi_2(x)} (\psi_1 - \psi_2) \, dy \right\} dx. \end{aligned}$$

Example 7.10 Find the volume within the cylinder $x^2 + y^2 = a^2$between the planes $y + z = b^2$ and $z = 0$.

Example 7.11 Find the volume of the solid bounded by the surface $z = 1 - 4x^2 - y^2$ and the plane $z = 0$.

7.5 Triple integrals

Here, we are intend to evaluate the integrals of the type

$$I = \iiint_v f(x, y, z)\, dxdydz$$

over the closed boundary v of the region. In particular, the volume is obtained by

$$V = \iiint_v dxdydz.$$

To evaluate, project onto coordinate plane, for example Oxy- plane, and then evaluate the double integral. For example, to evaluate $\iiint_v f(x, y, z)\, dxdydz$ over the sphere $x^2 + y^2 + z^2 = a^2$, we use

$$\int_v f\, dxdydz = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} f\, dzdydx.$$

Change of Variables

In the case of change of variables from the Cartesian coordinate system (x, y, z) to the curvilinear coordinate system (u, v, w), we use

$$\int_v f\, dxdydz = \int_v F(u, v, w)\, |J| dudvdw,$$

where $J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$ is the jacobian of the transformation.

Special cases

1. Cylindrical polar coordinates(ρ, ϕ, z)

$$x = \rho \cos\phi \quad y = \sin\phi, \quad z = z.$$
$$J = h_\rho h_\phi h_z = 1.\rho.1 = \rho.$$

2. Spherical polar coordinates(r, θ, ϕ)

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$
$$J = h_r h_\theta h_\phi = 1.r.r\sin\theta = r^2\sin^2\theta.$$

Example 7.12 Compute the integral $\int_v xyz\, dxdydz$ over a domain bounded by coordinate planes and the plane $x + y + z = 1$.

Example 7.13 Compute the volume of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Exercise: Deduce the volume of the sphere $x^2 + y^2 + z^2 = 9$.

Example 7.14 Compute $\iiint \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\, dxdydz$ over the region $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$.