SOUTH EASTERN UNIVERSITY OF SRI LANKA
Faculty of Applied Sciences
Department of Mathematical Sciences

MTC 12011 MATHEMATICS FOR BIOLOGY II

2019 / 2020

---

6. DIFFERENTIAL EQUATIONS

6.1 Introduction

Definition 6.1 An equation involving an independent variable, a dependent variable and derivatives of dependent variable with respect to the independent variable is called a ordinary differential equation.

Some examples of ordinary differential equations:

1. $x^2\left(\frac{d^2y}{dx^2}\right)^4 + 2x\left(\frac{dy}{dx}\right)^3 + y = x^2 + 3.$
2. $\frac{d^2x}{dt^2} + \omega^2 x = \sin x.$
3. $k\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}.$
4. $(x^2 + y^2)dx - 2xydy = 0.$
5. $\frac{d^2y}{dx^2} = \frac{W}{H}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$
6. $\left(\frac{d^3y}{dx^3}\right)^2 + 2\frac{d^2y}{dx^2}\frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^4 = 0.$

Definition 6.2 The order of the differential equation is the highest order of the derivatives that occurs in the equation.

Example 6.1 Identify the dependent variables, independent variables and the order of each of the differential equations in the above examples.

Definition 6.3 A relation which does not contain any derivatives such that this relation and the derivatives obtained from it is defined and satisfies the given differential equation is called the solution of the differential equation.

A solution which contains a number of independent arbitrary constants equal to the order of the differential equation is called the general solution or complete primitive.

Initial Value Problems (IVP)

Initial Conditions: Initial condition(s) is a (are set of) condition(s) on the solution at one point on the solution space that will allow us to determine which solution that we are after.

The number of initial conditions that are required for a given differential equation is the same as the order of the differential equation.

Initial Value Problem: An initial value problem (IVP) is a differential equation along with an appropriate number of initial conditions.

6.2 Separable Equations

If the given differential equation can be rewritten in the form $F(x)dx + G(y)dy = 0$, we say that variables are separable and the solution is obtained by

$$\int F(x)dx + \int G(y)dy = C,$$

where C is an arbitrary constant.

Example 6.2 Solve each of the following equations:

(i) $2x(1+y)\dfrac{dy}{dx} + (1+x^2)y = 0.$

(ii) $xy\,dy = (y+1)(1-x)dx = 0.$

(ii) $(xy^2 + x)dx + (yx^2 + y)dy = 0.$

Example 6.3 Solve the following initial value problems:

(i) $\dfrac{dy}{dx} = \dfrac{4\sin(2x)}{y}, \qquad y(0) = 1.$

(ii) $\dfrac{dy}{dx} + y\,\dfrac{(1+2x^2)}{x}, \qquad y(1) = 2.$

Example 6.4 Find the integral curve which satisfies the differential equation

$$(y^2 + 1)dx + (x^2 + 1)dy = 0$$

and passes through the origin.

Example 6.5 The size of certain bacterial colony increases at a rate proportional to the size of the colony. Suppose the colony occupied an area of $0.25\ \text{cm}^2$ initially, and after 8 hours it occupied an area of $0.35\ \text{cm}^2$.

(a) Estimate the size of colony t hours after the initial measurement.

(b) What is the expected size of the colony after 12 hours.

Example 6.6 The rate at which a sample decays is proportional to the size of the sample at that time. The half-life of radium is 1600 years, i.e., it takes 1600 years for any quantity to decay in to half of its original size. If a sample contains 50g of radium, how long will it be until it contains 45g?

6.3 Homogeneous Equations: General form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

Substitution: $y = vx$.

Example 6.7 Solve each of the following equations:

(i) $(x^2 - 3y^2)dx + 2xy\,dy = 0.$

(ii) $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right)dy = 0.$

(iii) $\dfrac{dy}{dx} + \dfrac{y^2}{x^2} = \dfrac{y}{x}.$

EXERCISES

1. Solve the following separable differential equations:

(a) $(x^2+1)\dfrac{dy}{dx}=xy.$

(b) $\dfrac{dy}{dx}=x(y-1).$

(c) $\dfrac{dy}{dx}=2x(1-y)^2.$

(d) $\dfrac{dy}{dx}=2y+1.$

(e) $(x^2+1)\dfrac{dy}{dx}=xy.$

(f) $\dfrac{dy}{dx}=e^{x+y}.$

(g) $\dfrac{dy}{dx}+3xy-x=0.$

2. Solve the following initial value problems:

(a) $\dfrac{dy}{dx}=x^4y^2;\ y(1)=-2.5$

(b) $\dfrac{dy}{dx}=y^2\left(\dfrac{1}{x}-x\right);\ y(1)=2.$

(c) $\dfrac{dy}{dx}=\dfrac{1}{y^2}\left(\dfrac{2}{x}-\dfrac{x}{2}\right);\ y(1)=1.$

(d) $\dfrac{dy}{dx}=\dfrac{x^2y-4y}{x+2};\ y(0)=1.$

3. The rate of change of a certain population is proportional to the square root of its size. Model this situation with a differential equation. If initial population is 16 and it doubles in 6 weeks, find the population after 20 weeks.

4 The rate of change of the velocity of an object is proportional to the square of the velocity. Model this situation with a differential equation. Initial velocity of a motor cycle is $10\ \text{kmh}^{-1}$ and after one seconds, it becomes $25\ \text{kmh}^{-1}$. If maximum velocity of the motor cycle is $140\ \text{kmh}^{-1}$, find the time taken to achieve it.

5. The amount of a certain medicine in the bloodstream decays exponentially with a half-life of 5 hours. In order to keep a patient safe during a one-hour procedure, there needs to be at least 50 mg of medicine per kg of body weight. How much medicine should be administered to a 60kg patient at the start of the procedure?

6. A cup of coffee is initially $170°\text{F}$ and is left in a room with room temperature $70°\text{F}$. The situation is governed by Newton's law of cooling:

$$\frac{dT}{dt}=k(T-T_{\text{env}}),$$

where $T(t)$ is the temperature of the coffee at time t and T_{env} is the room temperature. Suppose that the temperature reduced by 20 degree in one minutes. How long does it take for the coffee to cool to 110 degrees?

7. The population of fish in a pond is modeled by the differential equation

$$\frac{dP}{dt}=480-4P,$$

where time t is measured in years. Towards what number does the population of fish tend? If there are initially 10 fish in the pond, how long does it take for the number of fish to reach 90% of the eventual population (population as $t\to\infty$?)

8. Suppose it is known that the cells of a given bacterial culture divide every 3.5 hours (on average). If there are 500 cells in a dish to begin with, how many will there be after 12 hours?

9. Police arrived at the scene of a murder at 12.30 a.m. They immediately take and record the body's temperature, which is 90°F, and thoroughly inspect the area. By the time they finish the inspection, it is 2 a.m. They again take the temperature of the body which has dropped to 85°F. The temperature at the crime scene has reminded steady at $80°F$. The temperature T at time t of the body is governed by the Newton's law of cooling:

$$\frac{dT}{dt} = k(T_{env} - T)$$

where k is a positive constant and T_{env} is the temperature of the environment. Find the time of murder.

10. Solve the following homogeneous equations of first order:

(a) $xy\dfrac{dy}{dx} + 4x^2 + y^2 = 0,$

(b) $(x - y)dx + xdy = 0,$

(c) $(x - 2y)dx + 3xdy = 0,$

(d) $(x^2 - y^2)dx + 2xydy = 0,$

(e) $\left[x \sin\left(\dfrac{y}{x}\right) - y \cos\left(\dfrac{y}{x}\right)\right] dx + x \cos\left(\dfrac{y}{x}\right) dy = 0,$ Hint: $\displaystyle\int \tan z \, dz = -\ln|\cos z| + C$

11. Solve the initial value problem

$$2(x + 2y)dx + (y - x)dy = 0, \qquad y(1) = 0.$$

12. Solve the initial value problem

$$\frac{dy}{dx} - \frac{y}{x} = e^{\frac{y}{x}}, \qquad y(1) = 0.$$