SOUTH EASTERN UNIVERSITY OF SRI LANKA
Faculty of Applied Sciences
Department of Mathematical Sciences

MTM 22031 ELEMENTARY DIFFERENTIAL EQUATIONS

Course Content:

- Introduction to differential equations: Basic definitions, Classification of differential equations, Formation of ordinary differential equations, Solutions of a differential equation.
- First order first degree ordinary differential equations: Existence and uniqueness of solutions, Variable Separable equations, Homogeneous equations, Exact equations, Linear equations, Bernoulli's equation, Ricatti's equation, Clairaut's equation, Substitution methods.
- Applications.
- Approximation Method: Picard's method.

References:

1. Raisinghaniya M. D., *Ordinary and Partial Differential Equations*, S. Chand and company Ltd. New Delhi., 2008;
2. Zafar Ahsan, *Differential Equations and Their Applications,* PHI Learning Pvt. Ltd., 2004;
3. Dennis G. Zill, *A First Course in Differential Equations with Applications*, PWS Publishers., Boston, 1986.

CHAPTER 1 INTRODUCTION TO DIFFERENTIAL EQUATIONS

1.1 Introduction

Differential Equations: An equation involving the independent variables, dependent variables and derivatives of dependent variables with respect to independent variables.

Examples of Differential equations:

1. $x^2\left(\frac{d^2y}{dx^2}\right)^4+2x\left(\frac{dy}{dx}\right)^3+y=x^2+3.$
2. $\frac{d^2x}{dt^2}+\omega^2x=\sin x.$
3. $k\frac{d^2y}{dx^2}=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}.$
4. $y=x\frac{dy}{dx}+\frac{k}{\frac{dy}{dx}}.$
5. $\frac{d^2y}{dx^2}=\frac{W}{H}\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$
6. $\left(\frac{d^3y}{dx^3}\right)^2+2\frac{d^2y}{dx^2}\frac{dy}{dx}+x^2\left(\frac{dy}{dx}\right)^3=0.$
7. $\frac{\partial^3U}{\partial t^3}=k\left(\frac{\partial^2U}{\partial x^2}\right)^2.$
8. $(x^2+y^2)dx-2xydy=0.$

1.2 Classification of Differential Equations

Type Two main classes:

Ordinary differential equation (ODE): An equation containing only one independent variable and derivatives of dependent variables with respect to this independent variable.

Partial differential equation (PDE): An Equation that contains two or more independent variables partial derivatives and partial derivatives with respect to them.

In this course we deal with only ODEs.

Order: The order of the differential equation is the highest order of the derivatives that occurs in the equation.
Degree: The degree of the highest order derivative present in the equation, after the DE has been made free from the radicals and fractions as far as the derivative are concerned.

Linearity: A DE in which the dependent variables and all its derivatives present occur in the first degree only and no products of dependent variables and / or derivatives occur.

A DE which is not linear is called a <u>non-linear differential equation</u>.

Homogeneous: Each term of the equation contains the dependent variable.

A DE which is not homogeneous is called a <u>non-homogeneous (or inhomogeneous)</u> DE.

Example1 Classify the DEs given in section 1.1 as per type, order, degree and linearity. Also determine whether the equation is homogeneous or not.

Exercise (in class)

Classify each of the following DEs as per type, order, degree and linearity. Also determine whether the equation is homogeneous or not.

1. $2\,\dfrac{dy}{dx}+\dfrac{d^3y}{dx^3}=5\left(\dfrac{dy}{dx}\right)^2.$
2. $\dfrac{d^2y}{dx^2}+e^x=\tan y.$
3. $\dfrac{d^3y}{dx^3}=\left(1+\left(\dfrac{d^2y}{dx^2}\right)^2\right)^{5/2}.$
4. $\dfrac{\partial^2u}{\partial x^2}+\dfrac{\partial^2u}{\partial y^2}+\dfrac{\partial^2u}{\partial z^2}=0.$
5. $\left(\dfrac{dr}{ds}\right)^3=\left(\dfrac{d^4r}{ds^4}+1\right)^2.$
6. $\sqrt{\dfrac{d^2\rho}{d\theta^2}+\rho}=\sin\rho.$
7. $\dfrac{dy}{dx}=\left(\dfrac{1+x}{1+y}\right)^{1/3}.$
8. $\sin x\,\dfrac{d^2y}{dx^2}-(1-y^2)\dfrac{dy}{dx}+5y=0.$

1.3 Solution of a Differential Equation

Solution: A relation which does not contain any derivatives such that this relation and the derivatives obtained from it is defined and satisfies the given DE in some interval I.

Example 2 Show that $y(x) = \alpha\cos(nx + \beta)$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + n^2y = 0,$$

where α, β are arbitrary constants.

Example 3 Determine all values of r so that the differential equation

$$2\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 2y = 0$$

has a solution of the form $y = e^{rx}$.

Example 4 Show that $x^3 + 3xy^2 = 1$ gives a solution of the differential equation

$$2xy\frac{dy}{dx} + x^2 + y^2 = 0$$

on the interval $0 < x < 1$.

General Solution (*complete primitive*): A solution which contains a number of independent arbitrary constants equal to the order of the DE is called the general solution or complete primitive.

The family of solutions in this case is known as $n -$ parametric family.

The curves represented by this $n -$ parametric family is called the integral curve.

Particular Solution: A solution obtained from a general solution by giving particular values to one or more of the arbitrary constants is called the particular solution.

Example 5 Verify that $y = (x^2 + c)^2 + 1$ is a general solution of the differential equation

$$y' = 4x\sqrt{y - 1}.$$

Example 6 Verify that $y = e^{x^2}\int_0^x e^{-t^2}dt + e^{x^2}$ is a particular solution of the differential equation

$$\frac{dy}{dx} - 2xy - 1 = 0.$$

1.4 Formation of Differential Equations

Suppose we are given a primitive involving n independent arbitrary constants. Differentiating it successively n times and then eliminating n arbitrary constants from the above $(n + 1)$ equations, we get a DE of order n.

Example 7 Find the differential equation for which $r = c(1 + \cos\theta)$ is a solution.

Example 8 Find the differential equation for which $y(x) = c_1e^{-2x} + c_2e^{3x}$ where c_1, c_2 are arbitrary constants, is a complete primitive.

1.5 Initial Value Problems (IVP)

Initial Conditions: Initial condition(s) is a (are set of) condition(s) on the solution at one point on the solution space that will allow us to determine which solution that we are after. The number of initial conditions that are required for a given DE is the same as the order of the differential equation.

Initial Value Problem: An initial value problem (IVP) is a differential equation along with an appropriate number of initial conditions.

Example 9 Show that $y(x) = c_1 x^{-3/2} + c_2 x^{-1/2}$ is a solution to

$$4x^2 y'' + 12xy' + 3y = 0$$

for $x > 0$. Find the particular solution which satisfies the initial conditions $y(1) = 0, y'(1) = -1$.

CHAPTER 2 FIRST ORDER DIFFERENTIAL EQUATIONS

2.1 Separable Equations

If the given differential equation can be rewritten in the form $F(x)dx + G(y)dy = 0$, we say that variables are separable and the solution is obtained by

$$\int F(x)dx + \int G(y)dy = C,$$

where C is an arbitrary constant.

Example 2.1 Solve each of the following equations:

(i) $x^2(1+y)\dfrac{dy}{dx} + (1+x^2)y = 0.$ $\quad(ii)$ $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right).$

(iii) $x\,dx + (x^2+1)\cot y\; dy = 0.$ $\quad(iv)$ $\tan y \dfrac{dy}{dx} = \sin(x+y) + \sin(x-y).$

Example 2.2 Find the integral curve which satisfies the differential equation

$$(y^2+1)dx + (x^2+1)dy = 0$$

and passes through the origin.

Equations Reducible to Separable Form

1. Homogeneous Equations: General form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

Substitution: $y = vx$.

Example 2.3 Solve each of the following equations:

(i) $(x^2 - 3y^2)dx + 2xy\,dy = 0.$ $\quad(ii)$ $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right)dy = 0.$

2. Equations Reducible to Homogeneous Form: Equations of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}; \quad ab' - ba' \neq 0.$$

Substitution $x = u + h,\ y = v + k,$

where $u,\ v$ are new variables and $h,\ k$ are constants to be chosen.

Example 2.4 Solve

$$\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}.$$

3. Equations of the form

$$\frac{dy}{dx} = f(ax + by).$$

Substitution: $ax + by = u.$

Example 2.5 Solve each of the following equations:

(i) $\frac{dy}{dx} = \frac{x - y + 3}{2x - 2y + 3}.$ (ii) $\frac{dy}{dx} = \cos(x + y).$

2.2 Linear Equations

A first order differential equation is a linear equation if it is, or can be written, in the form

$$\frac{dy}{dx} + P(x)\, y = Q(x)$$

where $P(x)$ and $Q(x)$ are continuous functions of x alone on some interval I.

Working Rule

Step 1. Identify the equation as linear and write it in the general form $\frac{dy}{dx} + P(x)\, y = Q(x)$.

Step 2. Calculate $R(x) = \int P(x)dx$ omitting integrating constant and form $e^{R(x)}$.

Step 3. Multiply the equation by $e^{R(x)}$ and simplify to obtain

$$\frac{d}{dx}\left(e^{R(x)}\, y\right) = e^{R(x)} Q(x)$$

Step 4. Solve the equation.

Integrating Factor: The multiplication factor $e^{R(x)}$ is called an integrating factor (I.F.)

Example 2.6 Solve each of the following equations:

(i) $ydx - xdy + \ln x\, dx = 0.$ (ii) $\sin x \frac{dy}{dx} + y = \cos x.$

(iii) $y^2 dx + (3xy - 1)dy - 0.$

Equations Reducible to Linear Form

1. An equation of the form

$$f'(y)\,\frac{dy}{dx}+P(x)f(y)=Q(x).$$

Substitution $f(y)=v$.

Example 2.7 Solve each of the following equations:

$(i)\quad \dfrac{dy}{dx}+1=e^{x-y}.\qquad\qquad (ii)\quad \dfrac{dy}{dx}+x\sin 2y=x^3\cos^2 y.$

2. Bernoulli Equations

An equation of the form

$$\frac{dy}{dx}+P(x)y=Q(x)y^n,$$

where $n\neq 0,1$ is a constant, is called a Bernoulli equation.

Note:-

1. If $n=0$, the given equation is a linear equation.
2. If $n=1$, the equation is in the separable variable form

Solving Procedure:

Multiply the given equation by $(1-n)y^{-n}$ and get

$$(1-n)y^{-n}\frac{dy}{dx}+(1-n)P(x)y^{1-n}=(1-n)Q(x).$$

Now, substitute $v=y^{1-n}$.
Then, the equation reduces to the linear form.

Example 2.8 Solve each of the following equations:

$(i)\quad \dfrac{dy}{dx}+y=y^3x.\qquad\qquad (ii)\quad \cos\theta\,\dfrac{dr}{d\theta}-r\sin\theta=-r^2.$

2.3 Exact Equations

The differential $M(x,y)dx+N(x,y)dy$ is said to be exact if there exist a function $f(x,y)$ such that

$$M(x,y)dx+N(x,y)dy=df(x,y).$$

In this case, the differential equation

$$M(x,y)dx+N(x,y)dy=0$$

is called an exact differential equation.

Note:- $y^2dx + 2xydy = 0$ is an exact differential equation, for $y^2dx + 2xydy = d(y^2x)$ is exact. But, $ydx + 2xdy = 0$ is not exact.

Theorem 2.1
The differential equation $M(x, y)dx + N(x, y)dy = 0$ is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Proof. Omitted.

Solving exact differential equations

1. Solutions by inspection

Remember the following exact differentials for direct applications:

1. $d(xy) = xdy + ydx$
2. $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$
3. $d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{xdy - ydx}{x^2 + y^2}$
4. $d\left(\log\left(\frac{y}{x}\right)\right) = \frac{xdy - ydx}{xy}$
5. $d\left(\frac{e^x}{y}\right) = \frac{e^x(ydx - dy)}{y^2}$
6. $d\left(-\frac{1}{xy}\right) = \frac{xdy + ydx}{x^2y^2}$
7. $d(\log(x^2 + y^2)) = \frac{2xdy + 2ydx}{x^2 + y^2}$

Example 2.9 Solve each of the following exact equations by inspection:

(i) $(xe^{xy} + 2y)dy + ye^{xy}dx = 0.$ (ii) $2(u^2 + uv)du + (u^2 + v^2)dv = 0.$

(iii) $\left(y + \cos y + \frac{1}{2\sqrt{x}}\right)dx + (x - x\sin y - 1)dy = 0.$

2. General Working Rule

Step 1. Verify the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Step 2. Integrate M with respect to x, treating y as a constant.

Step 3. Integrate N with respect to y and exclude the terms which obtained from step 2.

Step 4. Add the two expressions obtained in step 2 and step 3 by considering the common terms, if any, only once and equate the result to an arbitrary constant, which is the required solution.

Example 2.10 Solve the equation

$$\frac{dy}{dx} + \frac{ax + hy + g}{hx + by + f} = 0$$

Example 2.11 Verify for exactness and solve $y\sin 2x \ dx - (y^2 + \cos^2 x)dy = 0.$

Equations reducible to Exact Form

Now we consider some differential equations of the form $Mdx + Ndy = 0$ that are not exact but can be made exact by multiplying the equation by a function called integrating factor.

Example 2.12 Show that

$$(x^3e^x - my^2)dx + mxy\,dy = 0$$

is not exact. Show also that x^{-3} is an integrating factor an hence solve the equation.

Example 2.13 Solve the equation

$$(x^3 + xy^2 + a^2y)dx + (y^3 + yx^2 - a^2x)dy = 0.$$

Applications

Example 2.14. A certain radioactive material is decaying at a rate proportional to the amount present. If a sample of 50 grams of the material was present initially and after 2 hours the sample lost 10% of its mass, find:
(a) An expression for the mass of the material remaining at any time t.
(b) The mass of the material after 4 hours.
(c) The half-life of the material.

Example 2.15 The body of a murder victim was discovered at 11.00 p.m. The doctor took the temperature of the body at 11.30 p.m., which was $94.6°F$. He again took the temperature after one hour when it showed $93.4°F$, and noticed that the room temperature of the room was $70°F$. Assuming normal temperature of human body is $98.6°F$, estimate the time of death.

3. EXISTENCE AND UNIQUENESS THEOREM

3.1.1 Picard's iteration

Theorem 3.1 Consider the IVP

$$\frac{dy}{dx} = f(x,y), \qquad y(x_0) = y_0 \tag{1}$$

Define a rectangle $R = \{(x,y): |x - x_0| \leq a,\ |y - y_0\} \leq b\}$ such that f is continuous in R. Then, $y = y(x)$ is a solution of the IVP if and only if $y(x)$ is a continuous solution of the integral equation

$$y = y_0 + \int_{x_0}^{x} f\big(s, y(s)\big)\,ds \tag{2}$$

Now, we shall list the procedure to find an approximation solution for the IVP. The method is due to Picard and therefore known as Picard's method of successive approximations.

Step 1. Select the initial guess as $y_0(x) = y_0$.

Step 2. Find at least three approximations $y_1(x),\ y_2(x),\ y_3(x)$ from the iteration equation

$$y_n(x) = y_0 + \int_{x_0}^{x} f\big(s, y_{n-1}(s)\big)\, ds, \qquad n = 1, 2, 3, \cdots.$$

Step 3. Guess the expression, if possible, for the n^{th} approximation $y_n(x)$.

Step 4. Taking the limit of $y_n(x)$ as $n \to \infty$, if the limit exist, we can find the exact solution of the IVP.

Example 3.1 Compute the first three Picard's iterates for the IVP

$$\frac{dy}{ds} = x^2 + y^2, \qquad y(0) = 0.$$

Example 3.2 Consider the initial value problem

$$\frac{dy}{dx} = 2x(y+1), \quad y(0) = 0.$$

(a) Find the first three Picard's approximations of the problem.
(b) Show that they converge to $y(x) = e^{x^2} - 1$.
(c) Use the above results to find a three decimal approximation for $e^{0.01}$.

The method has a simple expansion for the system of first order IVPs. We shall explain the expansion method using an example.

Example 3.3 Construct Picard iterates for the system of IVPs

$$\frac{dy}{dx} = x + z, \qquad \frac{dz}{dx} = z - y$$

where $y = 0$ and $z = 1$ when $x = 0$.

3.2 Picard's Existence and Uniqueness Theorem

Theorem 3.2 Consider the IVP

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

Define $R = \{(x, y) \colon |x - x_0| \le a,\ |y - y_0\} \le b\}$ such that f and $\frac{\partial f}{\partial y}$ are continuous in R. Let $M = \max_{(x,y)\in R} |f(x, y)|$ and let $h = \min\left(a, \frac{b}{M}\right)$. Then, the IVP has a unique solution $y = y(x)$ in the interval $|x - x_0| \le h$ and on the interval, $y(x)$ is such that $|y(x)| \le b$.

Remark: (i) If at least the condition 'f is continuous' is satisfied, then existence of the solution guaranteed in an interval possibly smaller than $|x - x_0| < a$ (See theorem 1).

(ii) The condition '$\frac{\partial f}{\partial y}$ is continuous' can be replaced by a weaker condition, known as Lipschits condition, for the uniqueness part. We omit this type of problems for this course.

Example 3.4 For the IVP

$$\frac{dy}{dx} = (y+1)\cos(x^2 y)\,, \qquad y(2) = -1$$

discuss the existence and uniqueness of a solution.

Example 3.5 For the IVP

$$\frac{dy}{dx} = \frac{2x+1}{y-1}, \qquad y(0) = -1$$

discuss the existence and uniqueness of a solution.

Example 3.6 For the IVP

$$\frac{dy}{dx} = x^2 + y^2, \qquad y(0) = 0$$

find the largest interval on which Picard's theorem guarantees the existence of a unique solution.

Remark: (i) The actual interval on which the solution exist may be larger than Picard's theorem guarantees.

(ii) The conditions stated in Picard's theorem are sufficient but not necessary. If the conditions do not hold, then the IVP may have either

(a) no solution,
(b) more than one solution, or
(c) a unique solution.

Example 3.7 Verify that $y_1(x) = 0$ and $y_2(x) = (2x)^{3/2}$ are two solutions of the IVP

$$\frac{dy}{dx} = 3y^{1/3}, \qquad y(0) = 0.$$

"The existence of two solutions contradicts the uniqueness part of the Picard's theorem." Do you agree with this statement? Give reasons.

Example 3.8 Show that the IVP

$$\frac{dy}{dx} = e^{-x^2} + y^4, \qquad y(0) = 0$$

has a unique solution on the interval $\left[0, \frac{1}{2}\right]$.

(You may assume that $e^{-t^2} \leq 1$ for each $t \in \mathbb{R}$).