SOUTH EASTERN UNIVERSITY OF SRI LANKA
Faculty of Applied Sciences
Department of Mathematical Sciences

MTM 22031 ELEMENTARY DIFFERENTIAL EQUATIONS
WORKSHEET

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1. Classify each of the following equations as per type (i.e., ordinary or partial), order, degree and linearity. Determine also whether it is homogeneous or not.

   $(a)\ \dfrac{d^2y}{dx^2} - 2y\dfrac{dy}{dx} + xy^2 = \dfrac{d^3}{dx^3}(e^{-2x}).$

   $(b)\ \dfrac{d^2y}{dx^2} + \sqrt{\dfrac{dy}{dx}} + xy = 0.$

   $(c)\ \dfrac{d^3y}{dx^3} + 4x\left(\dfrac{dy}{dx}\right)^2 = y\left(\dfrac{d^2y}{dx^2}\right) + e^y.$

2. For what values of the constant m will $y = e^{mx}$ be a solution of the differential equation
$$2y''' + y'' - 5y' + 2y = 0.$$
What is the general solution of the equation?

3. Verify and reconcile that $y = \ln x + A$ and $\sinh y + \cosh y = Cx$ are primitives of
$$\frac{dy}{dx} = \frac{1}{x}.$$

4. Given the second order differential equation $(x+1)y'' + xy' - y = (x+1)^2$; Determine whether or not each of the following function is a solution.

   $(a)\ y_1(x) = e^{-x} + x^2 + 1.$

   $(b)\ y_2(x) = x^2 + 1.$

5. Verify that $x^2 = 2y^2 \ln y$ is a solution of the differential equation
$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2}.$$

6. Verify that $x^2 + xy = C$ is a solution of the differential equation
$$x^2\frac{d^2y}{dx^2} - 2(x+y) = 0$$
for any value of the constant C.

7. Verify that $y = 4\, e^{3x} \sin x$ is a solution of the differential equation
$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 10\, y = 0.$$

8. Form the differential equation of $y^2 = 4a\,(x+a)$, where a is the parameter.

   Classify your equation as per order, degree, linearity and homogeneous.

9. Establish a differential equation by eliminating arbitrary constants A and B from the relation $v = A + \frac{B}{r}$.

10. Find the differential equation whose general solution is $y = A\, e^{2x} + B\, e^{-x} + x$

11. The equation $x^2 + (y-h)^2 = r^2$, where h and r are arbitrary constants, represents a family of circles of different radii and centres lying on the y – axis. Form a differential equation by eliminating h and r.

    Classify the obtained equation as per order, degree and linearity.

12. Find the differential equation of the one parameter family of functions

$$y = \frac{1}{Ce^x + 1}.$$

    Hence, solve the initial value problem $y' + y = y^2,\ y(1) = -1$.

13. Show that the function $y = x + 1 - e^x$ is a solution of the first order initial value problem

$$\frac{dy}{dx} = y - x, \qquad y(0) = 0.$$

14. Verify that $y(x) = \tan(x^3 + C)$, where C is an arbitrary constant is a solution of the differential equation

$$\frac{dy}{dx} = 3x^2(y^2 + 1).$$

    Find the integral curve passing through the point $(0, -1)$.

15. Find constants $a,\ b$ so that $y = (x+3)e^{2x}$ is a solution of the initial value problem

$$\frac{dy}{dx} = ay + e^{2x}, \qquad y(0) = b.$$

16. Solve each of the following first order differential equations:

    (a) $(1 + y^2)\sec^2 x\, dx + 2y \tan x\, dy = 0.$

    (b) $(1 + 4y^2)\sin 2x\, dx + \sin^2 x\, dy = 0.$

    (c) $3y \ln y\, dx + xdy = 0.$

    (d) $(y + 1)dy = (xy - x)dx.$

    (e) $(xy^2 + x)dx + (yx^2 + y)dy = 0.$

    (f) $\cot y\, dx + \sqrt{1 - x^2}dy = 0.$

    (g) $\dfrac{dy}{dx} + x^2 = x^2 e^{3y}.$

    (h) $\dfrac{dy}{dx} = xy + x + y + 1.$

    (i) $y\sqrt{1 + x^2 + y^2 + x^2y^2}\dfrac{dy}{dx} + x = 0.$

    (j) $\dfrac{dy}{dx} = \dfrac{\sin^2 y}{1 - x^2}.$

17. Using suitable substitutions, solve the following differential equations:

(a) $\dfrac{dy}{dy} = \dfrac{x - 2y + 3}{3x - 6y + 5}.$

(b) $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\left(\dfrac{y}{x}\right).$

(c) $\dfrac{dy}{dx} = 1 + \dfrac{y}{x} + \dfrac{y^2}{x^2}.$

(d) $\dfrac{dy}{dx} = 4x + y + 1;\ \ y(0) = 1.$

(e) $\dfrac{dy}{dx} + 1 = e^{\cos(x+y)} \sin(x + y).$

(f) $\dfrac{4x + 6y + 5}{3y + 2x + 4}\dfrac{dy}{dx} = 1.$

(g) $\dfrac{dy}{dx} = \sin(x + y) + \cos(x + y)$

(h) $\dfrac{dy}{dx} = \sin^2(x - y + 1).$

(i) $\dfrac{x + y - a}{x + y - b}\,\dfrac{dy}{dx} = \dfrac{x + y + a}{x + y + b}.$

(j) $x\dfrac{dy}{dx} + \dfrac{y^2}{x} = y.$

(k) $(xy^2 - x^2y)dx - x^3dy = 0.$

(l) $\dfrac{dy}{dx} = \dfrac{2y + 6}{x + y + 1}.$

18. Find the integral curve of the differential equation $3xy^2dy = (3y^3 - x^3)\,dx$ passing through the point $(1,2)$.

19. Let R, L and E be constants in the equation

$$iR + L\frac{di}{dt} = E,$$

which arise in an electrical circuit. Solve the equation if $i(0) = 0$.

20. A curve is defined by the condition that at each of its points (x, y), the slope is equal to twice the sum of the coordinates of the point. Find the curve.

21. The size of certain bacterial colony increases at a rate proportional to the size of the colony. Suppose the colony occupied an area of $0.25\ \text{cm}^2$ initially, and after 8 hours it occupied an area of $0.35\ \text{cm}^2$.

(a) Estimate the size of colony t hours after the initial measurement.

(b) What is the expected size of the colony after 12 hours.

22. Police arrived at the scene of a murder at 12.30 a.m. They immediately take and record the body's temperature, which is $90°\text{F}$, and thoroughly inspect the area. By the time they finish the inspection, it is 2 a.m. They again take the temperature of the body which has dropped to 85°F. The temperature at the crime scene has reminded steady at $80°F$. The temperature T at time t of the body is governed by the Newton's law of cooling:

$$\frac{dT}{dt} = k(T_{env} - T)$$

where k is a positive constant and T_{env} is the temperature of the environment. Find the time of murder.

23. The rate at which a sample decays is proportional to the size of the sample at that time. The half-life of radium is 1600 years, i.e., it takes 1600 years for any quantity to decay in to half of its original size. If a sample contains 50g of radium, how long will it be until it contains 45g?

24. Consider the differential equation

$$a\frac{dx}{dt} + bx = ke^{-\lambda t},$$

where a, b and k are positive constants and $\lambda\left(\neq \frac{b}{a}\right)$ is a non-negative constant. Solve the equation and discuss the long term behavior (what happens when $t \to \infty$) of the solution.

25. The current i flowing in an electrical circuit is governed by the equation

$$2\frac{di}{dt} + i = \cos 2t.$$

Find the current at any time assuming that initially it is zero.

26. Consider the initial value problem

$$\frac{dy}{dx} + 2xy = 6x, \qquad y(0) = 4.$$

(a) Solve the equation by the method of integrating factor.

(b) Reduce the equation in to separable variable form with the substitution $z = y - 3$ and hence solve.

27. Consider the initial value problem

$$\frac{dy}{dx} = 4x + y + 1; \;\; y(0) = 1.$$

(a) Using the substitution $4x + y + 1 = t$, reduce it into separable variable form and hence solve.

(b) Solve as a linear equation (by the integrating factor method).

28. Solve the following differential equations:

$(a)\ \ x\dfrac{dy}{dx}+2y=\dfrac{e^{3x}}{x}.$ $\quad$ $(d)\ \ x\dfrac{dy}{dx}-\dfrac{3}{x}=\dfrac{2}{y},\ \ y(2)=5.$

$(b)\ \ x\ \dfrac{dy}{dx}+2y=x^2\log x.$ $\quad$ $(d)\ \ (1+x^2)\ \dfrac{dy}{dx}+4xy=\dfrac{1}{(1+x^2)^2}.$

$(c)\ \ \dfrac{dy}{dx}=2xy+xe^{2x^2}.$ $\quad$ $(f)\ \ \left(\dfrac{e^{-2\sqrt{x}}-y}{\sqrt{x}}\right)\dfrac{dx}{dy}=1\,.$

29. Reduce each of the following equations into linear form using suitable substitution and hence solve:

$(a)\ \ \dfrac{dy}{dx}=\dfrac{y}{x}-y^2,\ \ \ y(1)=1.$ $\quad$ $(b)\ \ (x^3y^2+xy)dx=dy.$

$(c)\ \ \dfrac{dy}{dx}=\dfrac{x^2+y^2+1}{2xy}.$ $\quad$ $(d)\ \ \dfrac{dy}{dx}-\dfrac{\tan y}{1+x}=(1+x)e^x\sec y.$

$(e)\ \ 2y\ \dfrac{dy}{dx}+(1+y^2)\ln(1+y^2)=2x(1+y^2).$

30. The equation $\frac{dy}{dx}=A(x)\,y^2+B(x)\,y+C(x)$ is called Riccati's equation.

(a) Show that if $A(x)\equiv 0$, then the above equation is a linear equation, whereas if $C(x)\equiv 0$, then it is a Bernoulli equation.

(b) Show that if $y=f(x)$ is any solution of Riccati's equation, then the transformation $y=f+\frac{1}{v}$ reduces Riccati's equation to a linear equation in v.

(c) Verify that $f(x)=1$ is a solution of the Riccati's equation $\frac{dy}{dx}=(1-x)y^2+(2x-1)y-x$ and use the result of (b) to find the integral curve passing through the origin.

31. Check for the exactness of the following differential equation and then solve it.

$(a)\ \ (ye^{xy}+1)\,dx+(x\,e^{xy}+2)\,dy=0.$

$(b)\ \ (e^{x+y}+ay)dy+(e^{x+y}+bx^2)dx=0$

$(b)\ \ 2x\log y\,dx+\left(\dfrac{x^2}{y}+3y^2\right)dy=0$

32. Find the values of m and n for which $(xy^n+x^2)\,dx+(x^2y^m+y^3)\,dy=0$ is exact. Solve the equation for those values of m and n.

33. Find the value for the constant n for which the equation

$$(e^x\sin y+nx^2y^2)dx+(e^x\cos y+x^3y)dy=0$$

is exact. Solve the equation for this value of n.

34. Show that the equation $y\,dx+(2xy-e^{-2y})dy=0$ is not exact. Show further that $\mu(x,y)=e^{2y}/y$ is an integrating factor to this equation and hence solve.

35. Show that $(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0$ is not exact. Show also that $\frac{1}{x^2y^2}$ is an integrating factor of this equation and hence solve.

36. Show that the differential equation $(4x+3y^2)dx + 2xydy = 0$ is not exact.

Find an integrating factor of the form x^n, where n is a positive integer, and hence solve the given equation.

37. Consider the initial value problem

$$\frac{dy}{dx} = 1 - x - y, \qquad y(0) = 2.$$

(a) Using Picard's method, find the third approximation of the problem and use the result to approximate $y(0.2)$.

(b) Show that $y(x) = 2 - x$ is the exact solution of the IVP and use it to approximate $y(0.2)$.

(c) Find the error of the estimation at the point $x = 0.2$.

38. Apply Picard's iteration method to find the second approximation of the system of initial value problem

$$\frac{dy}{dx} = -y - 2z, \qquad y_0 = -1$$

$$\frac{dz}{dx} = 3y + 4z, \qquad z_0 = 2.$$

39. Convert the equation $y'' + 2y' - 3y = 0$, $y(0) = 1$, $y'(0) = 1$ into system of first order equations. Hence, find a third approximation for the equation. [Hint: Take $y'(x) = z(x)$.]

40. Find the largest interval on which Picard's theorem guarantees the existence of a unique solution for the initial value problem

$$\frac{dy}{dx} = -y^2, \qquad y(1) = 1.$$

Compare this interval with the solution of the actual interval.

41. Verify that $y_1(x) = 1 - x$ and $y_2(x) = -\frac{x^2}{4}$ are two solutions of the initial value problem

$$\frac{dy}{dx} = \frac{-x + \sqrt{x^2 + 4y}}{2}, \qquad y(2) = -1.$$

Explain why the existence of two solutions of the given problem does not contradict the uniqueness part of the Picard's theorem.

42. Show that Picard's theorem ensures a unique solution in the interval $|x| \le \frac{1}{8\sqrt{e}}$ for the IVP

$$\frac{dy}{dx} = (e^{-x^2} + 4y)e^{2y}, \quad y(0) = 0.$$

43. Prove that the initial value problem

$$\frac{dy}{dx} = \left(e^{x^2 - y^2}\right)(2 + y), \quad y(1) = -2$$

has only one solution.

What is this solution? Justify your answer.