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HMM 12222 DIFFERENTIAL GEOMETRY
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PART A: DIFFERENTIAL GEOMETRY OF CURVES

Course content : Introduction, Parameterized curves, Arc-length, Re-parameterization of a curve; Moving triads of lines and planes. Serret-Frenet formulae; Osculating circle, Involutes and evolutes, Helices.

1 PARAMETRIZED CURVES

1.1 Parameterization of Curves:

Definition 1.1 Let $[a, b]$ be an interval in the real axis $\mathbb{R}$. A *parameterized curve* in the Euclidean space $\mathbb{R}^3$ is a mapping $\underline{r}: [a, b] \longrightarrow \mathbb{R}^3$, where

$$\underline{r}(u) = \big(x(u), y(u), z(u)\big), \qquad u \in [a, b].$$

or, in short, $\underline{r}(u) = \big(x(u), y(u), z(u)\big)$, where u is a parameter.

Definition 1.2 A vector function $\underline{r}: [a, b] \to \mathbb{R}^3$, $\underline{r}(u) = \big(x(u), y(u), z(u)\big)$ is called *smooth* (C^∞) if the coordinate functions $x(u), y(u)$ and $z(u)$ are infinitely many times differentiable on the open interval (a, b) and continuous on $[a, b]$.

For $u \in (a, b)$, its *derivative* $\underline{r}': [a, b] \to \mathbb{R}^3$ is given by $\underline{r}'(u) = \big(x'(u), y'(u), z'(u)\big)$, i.e., by the derivatives of the coordinate functions.

Definition 1.3 The parameterized curve is *regular* if and only if its *tangent vector* (or *velocity vector*)

$$\underline{v}(u) = \underline{r}'(u) = \big(x'(u), y'(u), z'(u)\big) \neq (0, 0, 0)$$

for all $u \in [a, b]$. In other words, the curve is regular if

$$\left|\underline{r}'(u)\right| \neq 0.$$

Remark: the tangent vector $\underline{r}'(u)$ gives the *direction* of movement and $\left|\underline{r}'(u)\right|$ gives the *speed* of the movement (see section 2).

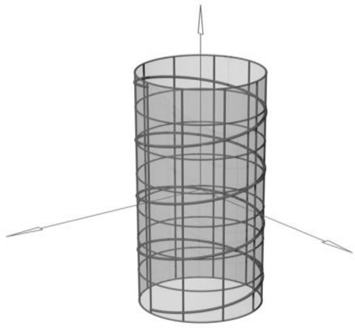
We shall assume, throughout this course, that the curves are smooth and regular.

Examples 1.1

1. Let $I = \mathbb{R}$ and $x(\lambda) = x_0 + \lambda\, l$, $y(u) = y_0 + \lambda m$, $z(u) = z_0 + \lambda\, n$. This represents a straight line passing through $(x_0,\ y_0,\ z_0)$ and parallel to the vector (l, m, n).

2. Let $I = [0, 2\pi]$ and $x(u) = a\cos u$, $y(u) = a\sin u$, $z(u) = 0$. This is a circle centered at the origin and has the radius a which lies in the Oxy plane.

3. Let $I = \mathbb{R}$ and $x(u) = a\cos u$, $y(u) = a\sin u$, $z(u) = bu$, $a, b \in \mathbb{R}$. This represents a *circular cylindrical helix*.



Note that $x^2 + y^2 = a^2$, $z = bu$.

That is, the curve lies on a circular cylinder and z rises with u at a constant rate.

Example 1.2 Parameterize the graph $y = f(x)$ on the plane $z = 0$.

Example 1.3 Find a parametric representation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

1.2 Re-parameterization of regular curves

A parameterization of a trace of a curve is not unique.

Example 1.4 The curves

$$\underline{r}(u) = (a\cos u, a\sin u, 0), \qquad u \in [0, 2\pi]$$

and the curve

$$\underline{r}(\theta) = (a\cos 2\theta, a\sin 2\theta, 0), \qquad \theta \in [0, \pi]$$

represents the same *trace* (circle). Here, the parameters u and θ are related by the equation $u = 2\theta$. These representations has same direction but different speed.

Definition 1.4 We say that a parameterized curve $\underline{r} = \underline{r}(u)$ is <u>naturally parameterized</u> if

$$\left|\underline{r}'(u)\right| = 1$$

for any $u \in I$. That is, the curve has unit speed. Usually, the natural parameter is denoted by s.

Definition 1.5 We define the arc length of a parameterized curve from a to u to be

$$l(a, u) = \int_a^u \left|\underline{r}'(t)\right| \, dt.$$

In particular, if $a = 0$, then the arc length is the natural parameter s. That is, the arc length is given by

$$s = \int_0^u \left|\underline{r}'(t)\right| \, dt.$$

Note: If $\underline{r}(u) = \big(x(u), y(u), z(u)\big)$, then

$$\frac{ds}{du} = \left|\underline{r}'(u)\right| = \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}$$

which gives $ds^2 = dx^2 + dy^2 + dz^2 = d\underline{r} \cdot d\underline{r} = |d\underline{r}|^2$. Thus,

$$\left|\frac{d\underline{r}}{ds}\right| = 1.$$

Example 1.5 Show that the parametrization of the circular helix given by

$$x = a\cos u\,, \qquad y = a\sin u\,, \qquad z = bu, \qquad u \in \mathbb{R}, \qquad a, b > 0$$

is regular and find the natural representation.
What is the arc length of the helix from $u = 0$ to $u = 2\pi$?

Example 1.6 Show that the curve

$$\underline{r} = \left(\frac{4}{5}\cos t\,, 1 - \sin t\,, -\frac{3}{5}\cos t\right)$$

is parameterized by arc-length (or has unit speed). Show further that this curve is the intersecting curve of the sphere $x^2 + (y-1)^2 + z^2 = 1$ and the plane $3x + 4z = 0$.

Exercises

1. Show that the curve defined by $\underline{r}(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}\,(1-t)^{3/2}, \frac{1}{\sqrt{2}}\,t\right)$ is regular and is naturally parameterized.

   Find the range of t such that the curve is smooth.

2. Show that the curve $\underline{r} = (\sin t\,, t, -\cos t)$ has constant speed.

   Reparametrize this curve by its arc length.

3. Consider the curve Γ given by $\underline{r} = (e^t\cos t\,, e^t\sin t\,, e^t)$. Show that Γ is regular.

   Show further that Γ lies entirely on the cone $x^2 + y^2 = z^2$.

   Find the natural parameterization of Γ.

4. Consider the curve C_1: $\underline{r}(u) = (u^2, \sin u\,, \cos u)$.

   Show that the curve C_1 is regular. Find the length of the curve C_1, when $u \in [0, 1]$.

2. MOVING TRIADS OF LINES AND PLANES

We can think of space curves as a path of a moving point. As the point moves along the curve, it is more convenient to consider a triad of mutually perpendicular right handed system of lines (local coordinate axes) which moves along with the point and the planes (local coordinate planes) spanned by a pair of them. In this section, we construct such important triads of lines and planes

2.1 Tangent

Definition 2.1 let $\underline{r} = \underline{r}(u)$ be the parameterized curve. The vector

$$\dot{\underline{r}} = \frac{d\underline{r}}{du}$$

is called the *tangent vector* (or *velocity vec*tor) of the curve. The straight line passing through $\underline{r}(u)$ and having the direction of the vector $\underline{\dot{r}}(u)$ is called the *tangent* to the curve at the point $\underline{r}(u)$ (or at the point u).

The vectorial equation to the tangent line is

$$\underline{R}(\lambda) = r(u) + \lambda\, \underline{\dot{r}}(u).$$

The unit vector $\underline{t}$ in this direction is called the *unit tangent vector* and is given by

$$\underline{t} = \frac{\underline{\dot{r}}(u)}{|\underline{\dot{r}}(u)|}.$$

Note: To avoid the confusion, here after we use $(\quad)'$ (prime) and $(\dot{\quad})$ (dot)to denote the derivatives with respect to the arc_length and any other parameter respectively.

Since $\left|\frac{d\underline{r}}{ds}\right| = 1$, $\frac{d\underline{r}}{ds}$ is the unit vector parallel to the tangent at the corresponding point. The sense of $\frac{d\underline{r}}{ds}$ is the same as that of the curve along with s increases. Thus, we have

$$\underline{t} = \frac{d\underline{r}}{ds} = \underline{r}'(s).$$

Remark:

$$\underline{t} = \frac{d\underline{r}}{ds} = \frac{d\underline{r}}{du} \cdot \frac{du}{ds} = \underline{\dot{r}}(u) \cdot \frac{du}{ds}.$$

Thus,

$$\frac{du}{ds} = \frac{1}{|\underline{\dot{r}}(u)|}$$

2.2 Principal normal

Since $\underline{t}$ is a unit vector, we have $\underline{t} \cdot \underline{t} = 1$. That is, $\underline{r}' \cdot \underline{r}' = 1$. Differentiating this with respect to the arclength s, we have

$$2\, \underline{r}' \cdot \underline{r}'' = 0.$$

i.e. the vector $\underline{r}''$ is perpendicular to the unit tangent vector, assuming $\underline{r}''(s) \not\equiv 0$.

Definition 2.2 The straight line passing through a point of the curve $\underline{r} = \underline{r}(s)$ and having the direction of the vector $\underline{r}''(s)$ is called the *Principal normal* to the curve at that point.
The vectorial equation to the principle normal is

$$\underline{R}(\lambda) = r(u) + \lambda\, \underline{r}''(u).$$

The unit vector in this direction is called the *unit principal normal vector* and is denoted by $\underline{n}$. That is,

$$\underline{n} = \frac{\underline{r}''}{|\underline{r}''|}.$$

Remark: The principal normal vector points the direction in which the curve is turning.

2.3 Binormal

Definition 2.3 The line perpendicular to both tangent and principal normal at a point of the curve is defined as binormal.
The unit vector along the binormal is denoted by $\underline{b}$ and the sense of $\underline{b}$ is chosen such that the vectors $\underline{t}$, $\underline{n}$, and $\underline{b}$, in that order, form a right handed system. That is,

$$\underline{b} = \underline{t} \times \underline{n}.$$

The vectorial equation of the binormal line has the form is $\underline{R}(\lambda) = \underline{r} + \lambda\, \underline{b}$.

Remark: The vectors $\underline{t}$, $\underline{n}$, and $\underline{b}$ are

(i) unit vectors, so that $\underline{t} \cdot \underline{t} = \underline{n} \cdot \underline{n} = \underline{b} \cdot \underline{b} = 1$.

(ii) mututually perpendicular, so that $\underline{t} \cdot \underline{n} = \underline{n} \cdot \underline{b} = \underline{b} \cdot \underline{t} = 0$.

(iii) form a right handed system, i.e. $\underline{t} \times \underline{n} = \underline{b}$, $\underline{n} \times \underline{b} = \underline{t}$, $\underline{b} \times \underline{t} = \underline{n}$.

2.4 Osculating Plane

Definition 2.4 The plane spanned by the tangent and the principal normal at a point on the curve is called the osculating plane at that point.
Equation of the osculating plane is

$$\left[(\underline{R} - \underline{r}),\ \underline{r}',\ \underline{r}''\right] = 0.$$

Example 2.1 Show that the equation of the osculating plane can also be given as

$$\left[(\underline{R} - \underline{r}),\ \underline{\dot{r}},\ \underline{\ddot{r}}\right] = 0.$$

In the determinant form, the equation becomes

$$\begin{vmatrix} X - x_0 & Y - y_0 & Z - z_0 \\ \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \\ \ddot{x}_0 & \ddot{y}_0 & \ddot{z}_0 \end{vmatrix} = 0.$$

Since the unit binormal vector is perpendicular to the osculating plane, the equation can also be written as

$$(\underline{R} - \underline{r}) \cdot \underline{b} = 0.$$

Theorem 2.1 If the parameterized curve is plane, then the osculating plane at each point of the curve coincides with the plane of the curve itself. Conversely, if the parameterized curve has the same osculating plane at each point of the curve, then the curve lies entirely on the osculating plane.

Example 2.2 Find the equation of the osculating plane at any point of the circle of radius a which lies on the plane $z = 5$.

2.5 Normal Plane

Definition 2.5 The *normal plane* at a point of the curve is the plane passes through that point and is perpendicular to the tangent line to the curve at that point.

The vectorial equation for the normal plane is $(\underline{R}-\underline{r})\cdot\dot{\underline{r}}=0$ or $(\underline{R}-\underline{r})\cdot\underline{t}=0$.

Remark: Since the binormal and principal normal are perpendicular to the tangent at any point of the curve, they span the normal plane. Thus, the equation of the normal plane can be given as

$$\underline{R}=\underline{r}+\lambda\,\underline{b}+\mu\,\underline{n} \quad \text{or} \quad \left[(\underline{R}-\underline{r}),\ \underline{b},\ \underline{n}\right]=0.$$

2.6 Rectifying Plane

Definition 2.6 The plane passes through a point of the curve and containing the tangent and the binormal at that point is called the *rectifying plane* to the curve at that point.

The vector equation for the rectifying plane is $\underline{R}=\underline{r}+\lambda\underline{t}+\mu\underline{b}$ or $\left[(\underline{R}-\underline{r}),\underline{t},\underline{b}\right]=0$

Remark: Since the tangent and binormal are perpendicular to the principal normal at any point of the curve, rectifying plane is perpendicular to the principal normal. Hence, we have

$$(\underline{R}-\underline{r})\cdot\underline{r}''=0 \quad \text{or} \quad (\underline{R}-\underline{r})\cdot\underline{n}=0.$$

Example 2.3 Find the equations of the tangent, principle normal, binormal, osculating plane, normal plane and rectifying plane at a general point of the circular helix

$$\underline{r}(\theta)=(a\cos\theta\,,a\sin\theta\,,b\theta), \qquad a,b>0.$$

Example 2.4 Consider the curve $\underline{r}(u)=(u^2,e^u,u+1)$, where u is a parameter. Find the equations of the tangent, normal plane, osculating plane, binormal, principle normal and rectifying plane at $u=1$.

EXERCISES 2

Find the equations of the tangent, normal plane, osculating plane, binormal, principle normal and rectifying plane for the following space curves at the given points:

1. $\underline{r}(\theta)=\left(-\frac{1}{\sqrt{2}}\sin\theta\,,\frac{1}{\sqrt{2}}\sin\theta\,,\cos\theta\right)$, at $\theta=\pi/2$.

2. $\underline{r}(t)=\left(\frac{1}{3}(1+t)^{3/2},\frac{1}{3}\,(1-t)^{3/2},\frac{1}{\sqrt{2}}\,t\right)$, at $t=0$.

3. $\underline{r}(u)=\left(\cos^2 u-\frac{1}{2},\ \sin u\cos u\,,\ \sin u\right)$, at $u=\pi/4$.

4. $\underline{r}(t)=(3\cosh 2t\,,3\sinh 2t\,,6t)$, at $t=0$.

3. SERRET – FRENET FRAME

3.1 Curvature

Definition 3.1 The arc-rate at which the tangent changes direction as a point P moves along the curve is called the curvature vector of the curve at P and is denoted by $\underline{\kappa}$.

$$\text{i.e.} \quad \underline{\kappa} = \frac{d\underline{t}}{ds}.$$

The magnitude of the curvature vector, κ, is called the curvature at P.

$$\text{i.e.} \quad \kappa = \left|\frac{d\underline{t}}{ds}\right| = \left|\frac{d}{ds}\left(\frac{d\underline{r}}{ds}\right)\right| = |\underline{r}''|.$$

When $\kappa \neq 0$, $\rho := 1/\kappa$ is called the radius of curvature at P.

Theorem 3.1 A curve is a straight line if and only if the curvature vanishes at all points of the curve. That is, $\underline{t}' = \underline{0}$.

3.2 Torsion

Definition 3.2 The magnitude of the arc-rate at which the binormal changes its direction as a point P moves along the curve is called the torsion (the second curvature) of the curve at P and is denoted by τ.

$$\text{i.e.} \quad \tau = \left|\frac{d\underline{b}}{ds}\right|.$$

When $\tau \neq 0$, $\sigma := 1/\tau$ is called the radius of torsion at P.

Remark 3.1: Curvature at a point of a curve is always positive where as torsion of the curve can be positive or negative.

Theorem 3.2 A curve is a plane curve if and only if the torsion vanishes at all points of the curve. That is, $\underline{b}' = \underline{0}$.

Example 3.1 In the usual notation, show that $\underline{b} = \frac{\dot{\underline{r}} \times \ddot{\underline{r}}}{|\dot{\underline{r}} \times \ddot{\underline{r}}|}$. Hence show that the curve $\underline{r} = (au + b, cu + d, u^2)$ is planar.

3.3 Serret – Frenet formulae

Theorem 3.3 In the usual notation, the curvature and the torsion at a point of a space curve is related with the corresponding unit vectors by

$$(i)\ \frac{d\underline{t}}{ds} = \kappa\,\underline{n}, \qquad (ii)\ \frac{d\underline{n}}{ds} = \tau\,\underline{b} - \kappa\,\underline{t}, \qquad (iii)\ \frac{d\underline{b}}{ds} = -\tau\,\underline{n}$$

Example 3.2 If the curvature and torsion of the curve $\underline{r} = (e^u \cos u\,, e^u \sin u\,, e^u)$ are κ and τ are respectively, show that $\kappa/\tau = \sqrt{2}$.

3.4 Expressions for curvature and torsion

Theorem 3.4 At any point of the curve $\underline{r} = \underline{r}(u)$, the curvature is given by

$$\kappa = \frac{|\dot{\underline{r}} \times \ddot{\underline{r}}|}{|\dot{\underline{r}}|^3}.$$

If the curvature $\kappa \neq 0$, the torsion is given by

$$\tau = \frac{|\dot{\underline{r}} \times \ddot{\underline{r}} \cdot \dddot{\underline{r}}|}{\kappa^2 |\dot{\underline{r}}|^6} = \frac{|\dot{\underline{r}} \times \ddot{\underline{r}} \cdot \dddot{\underline{r}}|}{|\dot{\underline{r}} \times \ddot{\underline{r}}|^2}.$$

Example 3.3 Find the radius of curvature at any point of the circle of radius a. Verify that the torsion vanishes identically for the plane curves.

3.5 Helices

Definition 3.3 A space curve is called a *helix* if its tangents make a constant angle with a fixed direction in space.

Verification: In example 5, the tangent

$$\underline{t}(\theta) = \frac{1}{\sqrt{a^2 + b^2}}(-a\sin\theta, a\cos\theta, b).$$

Note that $\underline{t} \cdot \underline{k} = \frac{1}{\sqrt{a^2+b^2}}$ =const.
i.e. $\underline{t}$ makes a constant angle with the Oz axis, which is known as the axis of the helix.

Theorem 3.5: Lancret's Theorem

A space curve with non-zero curvature is a helix if and only if the ratio between its torsion and its curvature is constant.

Example 3.4 Prove that the curve $\underline{r}(u) = (2u, 3u^2, -3u^3)$ is a helix.

EXERCISES 3

1. The serret – Frenet equations for a space curve are given as
$$\frac{d\underline{t}}{ds} = \kappa\, \underline{n}, \qquad \frac{d\underline{b}}{ds} = -\tau\, \underline{n}, \qquad \frac{d\underline{n}}{ds} = A\, \underline{t} + B\, \underline{b}.$$
By differentiating the quantities $\underline{t} \cdot \underline{n}$ and $\underline{b} \cdot \underline{n}$ with respect to the arc-length s or otherwise, find A and B in terms of κ and τ.

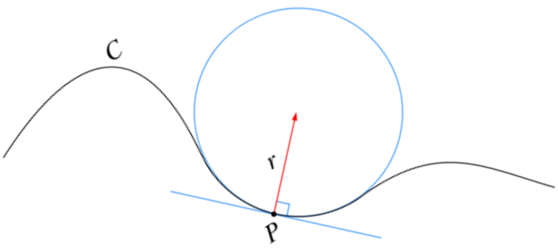
2. A curve Γ is defined parametrically by the equation $\underline{r}(u) = \left(\frac{u^3}{6}, \frac{u^2}{2}, u\right)$. If the curvature and torsion at any point of Γ is κ and τ respectively, show that
$$\kappa = |\tau| = \frac{4}{(u^2+2)^2}.$$

3. A twisted curve is given by the parametric equations
$$x = \tan u\,, \qquad y = \cot u\,, \qquad z = \sqrt{2}\ln(\tan u)\,;$$
u being a parameter. If $\kappa,\ \tau$ are the curvature and torsion at the point $'u'$ respectively, prove that
$$\kappa = |\tau| = \frac{\sqrt{2}\sin^2 2u}{4}.$$
Find the equation of the rectifying plane at the point $(1, 1, 0)$ on the curve.

4. A twisted curve Γ is given by the parametric equations
$$x = 2t + \sinh 2t\,, \qquad y = 2\sinh^2 t\,, \qquad z = 4\sinh t\,;$$
t being a parameter. If $\kappa,\ \tau$ are the curvature and torsion at the point $'t'$ respectively, prove that
$$\kappa = |\tau| = \frac{\operatorname{sech}^3 t}{8}.$$
Prove also that the lines through the origin and parallel to the binormal of Γ are generators of the cone $x^2 = y^2 + z^2$.

4. OSCULATING CIRCLE

Definition 4.1 The circle which touch a given curve at a point P whose radius is equal to radius of curvature and whose centre, known as the centre of curvature, is in the direction of unit principal normal vector is called the osculating circle at the point P.



Clearly, the centre of curvature is given by
$$\underline{c} = \underline{r} + \rho\,\underline{n}.$$

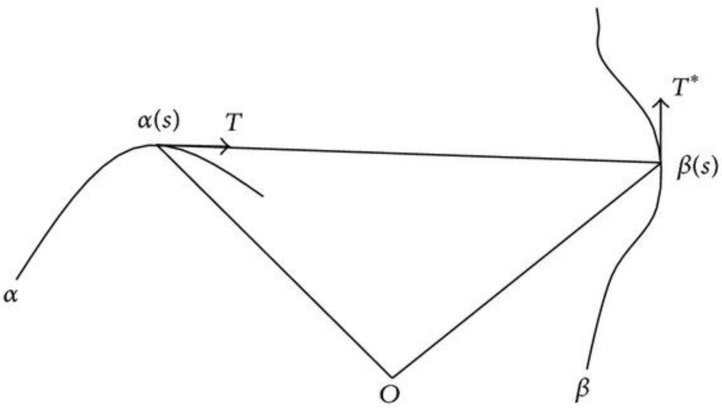
Locus of centres of curvature

Definition 4.2 Let Γ be a given curve and P be any point on Γ. As P moves along Γ, centr of curvature C traces out another curvature Γ_1: $\underline{r_1}(s) = \underline{r}(s) + \rho(s)\,\underline{n}(s)$, called the locus centre of curvature of Γ.

Example 4.1 Let Γ be a curve with constant curvature and Γ_1 be the locus of centres of curvature of Γ. Show that the curvature at any point of Γ_1 is the same as that of the corresponding point of Γ. Show further that the corresponding torsions of both curves vary inversely.

5. INVOLUTES AND EVOLUTES

Definition 5.1 If there is a one – to – one correspondence between the points of two curves Γ, Γ_1 such that the tangent at any point of Γ is a normal at the corresponding point of Γ_1, then Γ_1 is called an involute of Γ and Γ is called an evolute of Γ_1.



Example 5.1 Show that the equation of the involute at a point of a given curve $\underline{r} = \underline{r}(s)$ is given by

$$\underline{r_1} = \underline{r} + (c - s)\underline{t},$$

where c is a constant. Show further that the curvature κ_1 and the torsion τ_1 of the involute are given by

$$\kappa_1 = \frac{\sqrt{\rho^2 + \sigma^2}}{\sigma\,(c - s)}, \qquad \tau_1 = \frac{\rho(\sigma\rho' - \rho\sigma')}{(\rho^2 + \sigma^2)(c - s)}.$$

Remark: Since c is an arbitrary constant, there is an involute associate to each value of c. i.e. there are infinitely many involutes corresponding to each point of the given curve.

Example 5.2 Show that the equation of the evolute at a point of a given curve $\underline{r} = \underline{r}(s)$ is given by

$$\underline{r_1} = \underline{r} + \rho\underline{n} + \rho\cot\left(\int \tau ds + c\right)\underline{b},$$

where c is a constant.