CHAPTER 05

MODELING WITH HIGHER ORDER DIFFERENTIAL EQUATION

5.1 SPRING / MASS SYSTEM: FREE UNDAMPED MOTION

Hooke's Law

Suppose a mass m is coupled to the free end of a flexible spring that is hanging vertically from a rigid support. The amount of stretch, or elongation, of the spring will of course depend on the mass; masses with different weights stretch the spring by differing amounts. By Hooke's law the spring itself exerts a restoring force F opposite to the direction of elongation and proportional to the amount of elongation s. Simply stated, $F = ks$, where k is a constant of proportionality called the **spring constant.**

Example 01: If a mass weighing 10 pounds stretches a spring $\frac{1}{2}$ foot, find spring constant k?

After a mass m is attached to a spring, it stretches the spring by an amount s and attains a position of equilibrium at which its weight W is balanced by the restoring force ks. The condition of equilibrium is $mg = ks \ or \ mg - ks = 0$. If the mass is displayed by an amount x from its equilibrium position, the restoring force of the spring is then $k(x + s)$. Assuming that there are no retarding forces acting on the system and assuming that the mass vibrates free of other external forces free motion we can equate Newton's second law with the net, or resultant, force of the restoring force and the weight:

$$m\frac{d^2x}{dt^2} = -k(s + x) + mg = -kx + mg - ks = -kx \tag{1}$$

The negative sign in (1) indicates that the restoring force of the spring acts opposite to the direction of motion. Furthermore, we adopt the convention that displacements measured below the equilibrium position $x = 0$ are positive.

DE of Free Undamped Motion: By dividing (1) by the mass m, we obtain the second-order differential equation.

$$d^2x/dt^2 + (k/m)x = 0$$

or

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad (2)$$

Where $\omega^2 = \frac{k}{m}$. Equation (2) is said to describe **simple harmonic motion or free undamped motion.** Two obvious initial conditions associated with (2) are $x(0) = x_0 \quad and\ x'(0) = x_1$, the initial displacement and initial velocity of the mass, respectively. For example, if $x_0 > 0$, $x_1 < 0$, the mass starts from a point below the equilibrium position with an imparted upward velocity. When $x'(0) = 0$, the mass is said to be released from rest. For example, if $x_0 < 0$, $x_1 = 0$, the mass is released from rest from a point $|x_0|$ units above the equilibrium position.

Equation Motion: To solve equation (2) we note that the solutions of its auxiliary equation $m^2 + \omega^2 = 0$ are the complex numbers $m_1 = \omega i, m_2 = -\omega i$. We find the general solution of (2) to be

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t \qquad (3)$$

The **period** of motion described by (3) is $T = 2\pi/\omega$. The number T represents the time, it takes the mass to execute one cycle of motion. A cycle is one complete oscillation of the mass, that is, the mass m moving from, say, the lowest point below the equilibrium position to the point highest above the equilibrium position and then back to the lowest point. From the graphical viewpoint $T = 2\pi/\omega$ seconds is the length of the time interval between two successive maxima (or minima) of $x(t)$. Keep in mind that a maximum of $x(t)$ is a positive displacement corresponding to the mass attaining its greatest distance below the equilibrium position. Whereas a minimum of $x(t)$is negative displacement corresponding to the mass attaining its greatest height above the equilibrium position. We refer to either case as an **extreme displacement** of the mass. The **frequency** of motion is $f = 1/T = \omega/2\pi$ and is the number of cycles completed each second. For example, if $x(t) = 2\cos 3\pi t - 4\sin 3\pi t$, then the period is $T = 2\pi/3\pi = 2/3$ s, and the frequency is $f = 3/2$ cycles/s. From a graphical viewpoint the graph of $x(t)$ repeats every 2/3 second, that is, $x\left(t + \frac{2}{3}\right) = x(t)$, and $\frac{3}{2}$ cycles of the graph are completed each second (or, equivalently, three cycles of the graph are completed every 2 seconds). The number $\omega = \sqrt{k/m}$ (measured in radians per second) is called the **circular frequency** of the system. Depending on which text you read both $f = \omega/2\pi$ and ω are also referred to as the **natural frequency** of the system. Finally, when the initial conditions are used to determine the constants c_1 and $c_2$in (3), We say that the resulting particular solution or response is the **equation of motion.**

Example 02:

A mass weighing 2pounds stretches a spring 6 inches. At $t = 0$ the mass is released from a point 8 inches below the equilibrium position with an upward velocity of $4/3\frac{ft}{s}$. Determine the equation of motion.

5.2 SPRING / MASS SYSTEM : FREE DAMPED MOTION

The concept of free harmonic motion is somewhat unrealistic, since the motion described by equation (1) assumes that there are no retarding forces acting on the moving mass. Unless the mass is suspended in a perfect vacuum, there will be at least a resisting force due to the surrounding medium. The mass could be suspended in a viscous medium or connected to a dashpot damping device.

DE of free Damped Motion : In the study of mechanics, damping forces acting on a body are considered to be proportional to a power of the instantaneous velocity. In particular, we shall assume throughout the subsequent discussion that this force is given by a constant multiple of dx/dt. When no other external forces are impressed on the system, it follows from Newton's second law that

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}, \qquad (4)$$

Where β is a positive *damping constant* and the negative sign is a consequence of the fact that the damping force acts in a direction opposite to the motion. Dividing (4) by the mass m, we find that the differential equation of **free damped motion** is

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

Or

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0, \tag{5}$$

Where $2\lambda = \frac{\beta}{m}$, $\omega^2 = \frac{k}{m}$

The symbol 2λ is used only for algebraic convenience because the auxiliary equation is $m^2 + 2\lambda m + \omega^2 = 0$, and the corresponding roots are then

$m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2}$, $m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2}$,

We can now distinguish three possible cases depending on the algebraic sign of $\lambda^2 - \omega^2$. Since each solution contains the *damping* factor $e^{-\lambda t}$, λ> 0, the displacements of the mass become negligible as time t increases.

Case I: $\lambda^2 - \omega^2 > 0$ In this situation the system is said to be overdamped because the damping coefficient β is large when compared to the spring constant k. The corresponding solution of (11) is $x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$ or

$$x(t) = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2-\omega^2}t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}t}\right). \tag{6}$$

This equation represents a smooth and nonoscillatory motion.

Case II: $\lambda^2 - \omega^2 = 0$ The system is said to be critically damped because any slight decrease in the damping force would result in oscillatory motion. The general solution of (5) is $x(t) = c_1 e^{m_1 t} + c_2 t e^{m_2 t}$ or

$$x(t) = e^{-\lambda t}(c_1 + c_2 t) \tag{7}$$

Notice that the motion is quit similar to that of an overdamped system. It is also apparent from (14) that the mass can pass through the equilibrium position at most one time.

Case III: $\lambda^2 - \omega^2 < 0$ In this case the system is said to be underdamped since the damping coefficient is small in comparison to the spring constant. The root $m_1\ and m_2$ are now complex:

$m_1 = -\lambda + \sqrt{\omega^2 - \lambda^2}i$, $m_2 = -\lambda - \sqrt{\omega^2 - \lambda^2}i$,

Thus the general solution of equation (5) is

$$x(t) = e^{-\lambda t}\left(c_1 \cos\sqrt{\omega^2 - \lambda^2}\,t + c_2 \sin\sqrt{\omega^2 - \lambda^2}\,t\right).$$

The motion described by (15) is oscillatory: but because of the coefficient $e^{-\lambda t}$, the amplitudes of vibration → 0 as t→ ∞.

Example 03: Overdamped Motion

Verify that the following IVP

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 0 \quad x(0) = 1 \quad x'(0) = 1$$

Representing a overdamped motion and find the value of t which function has an extremum.

Example 04: Critically Damped Motion

A mass weighing 8 pounds stretches a spring 2 feet. Assuming that a damping force numerically equal to 2 times the instantaneous velocity acts on the system, determine the equation of motion if the mass is initially released from the equilibrium position with an upward velocity of 3 ft/s.

Example 06: Underdamped Motion

A mass weighing 16 pounds is attached to a 5-foot-long spring. At equilibrium the spring measure 8.2 feet. If the mass is initially released from rest at a point 2 feet above the equilibrium position, find the displacement $x(t)$ if it is further known that the surrounding medium offers a resistance numerically equal to the instantaneous velocity.

5.3 SPRING / MASS SYSTEM : DRIVEN MOTION

DE of Driven Motion with Damping : Suppose we now into consideration an external force $f(t)$ acting on a vibrating mass on a spring. For example, $f(t)$ could represent a driving force causing an oscillatory vertical motion of the support of the spring. The inclusion of $f(t)$ in the formulation of Newton's second law gives the differential equation of **driven** or **forced motion:**

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} + f(t). \qquad (1)$$

Dividing (1) by m gives

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t), \tag{2}$$

Where $F(t) = f(t)/m$ and, as in the preceding section, $2\lambda = \frac{\beta}{m}, \omega^2 = \frac{k}{m}$. To solve the latter nonhomogeneous equation, we can either the method of undetermined coefficients or variation of parameters.

Example 07:

Interpret and solve the following IVP

$$\frac{1}{5}\frac{d^2x}{dt^2} + 1.2\frac{dx}{dt} + 2x = 5\cos 4t \quad x(0) = \frac{1}{2} \quad x'(0) = 0$$

Transient and Steady-State Terms: When F is a periodic function, such as $F(t) = F_0 \sin\gamma t \ \ or \ \ F(t) = F_0\cos\gamma t,$ the general solution for $\lambda > 0$ is the sum of a nonperiodic function $x_c(t)$ and a periodic function $x_p(t)$. Moreover, $x_c(t)$ dies off as time increases $-$that is, $\lim_{t\to\infty} x_c(t) = 0$. Thus for large value of time, the displacements of the mass are closely approximated by the particular solution x_p. The complementary function $x_c(t)$ is said to be a **transient term** or **transient solution**, and the function $x_p(t)$, the part of the solution that remains after an interval of time, is called a **steady-state term** or **steady-state solution**.

DE of Driven Motion without Damping: with a periodic impressed force and no damping force, there is no transient term in the solution of a problem. Also, we shall see that a periodic impressed force with a frequency near or the same as the frequency of free undamped vibrations can cause a severe problem in any oscillatory mechanical system.

Pure Resonance: Although equation is not defined for $\gamma = \omega$, it is interesting to observe that its limiting value as $\gamma \to \omega$ can be obtained by applying L' Hospital's Rule. This limiting process is analogous to "tuning in" the frequency of the driving force $(\frac{\gamma}{2\pi})$ to the frequency of free vibrations $\left(\frac{\omega}{2\pi}\right)$. Intuitively, we expect that over a length of time we should be able to substantially increase the amplitudes of vibration.

5.4 SERIES CIRCUIT ANALOGUE

LCR-Series Circuits: As was mentioned in the introduction to this chapter, many different physical systems can be described by a linear second- order differential equation similar to the differential equation of forced motion with damping:

$$m\frac{d^2x}{dt^2} + kx + \beta\frac{dx}{dt} = f(t). \quad (1)$$

If $i(t)$ denotes current in the **LCR-series electrical circuit**, then the voltage drops across the inductor, resistor, and capacitor are as shown in figure. By Kirchhoff's second law the sum of these voltages equals the voltage $E(t)$ impressed on the circuit; that is,

$$L\frac{di}{dt} + Ri + \frac{1}{c}q = E(t) \quad (2)$$

But the charge $q(t)$ on the capacitor is related to the current $i(t)$ by $i = dq/dt$, so (2) becomes the linear second-order differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{c}q = E(t). \quad (3)$$

The nomenclature used in the analysis of circuits is similar to that used to describe spring / mass system.

If $E(t) = 0$, the **electrical vibrations** of the circuits are said to be free. Because the auxiliary equation for (3) is $Lm^2 + Rm + 1/c = 0$, there will be three forms of the solution with $R \neq 0$, depending on the value of the discriminant $R^2 - 4L/C$. We say that the circuit is

Overdamped if $R^2 - 4L/C > 0$,

Critically damped if $R^2 - 4L/C = 0$,

Underdamped if $R^2 - 4L/C < 0$.

In each of these three cases the general solution of (3) contains the factor $e^{-Rt/2L}$, so $q(t) \to 0$ as $t \to \infty$. In the underdamped case when $q(0) = q_0$, the charge on the capacitor oscillates as it decays; in other words, the capacitor is charging and discharging as $t \to \infty$. When $E(t) = 0$ and $R = 0$, the circuit is said to be undamped, and the electrical vibrations do not approach zero as t increases without bound; the response of the circuit is **simple harmonic.**

Example 08: Underdamped Series Circuit

Find the charge $q(t)$ on the capacitor in an *LRC* circuit when L=0.25 henry(h) $R = 10$ ohms, $c = 0.001$ faraad (f), $E(t) = 0$, $q(0) = q_0$ coulombs and $i(0) = 0$.