SOUTH EASTERN UNIVERSITY OF SRI LANKA

DEPARTMENT OF MATHEMATICAL SCIENCES

FACULTY OF APPLIED SCIENCES

MTS 00033 MULTIVARIATE CALCULUS

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1. LIMITS AND CONTINUITY

1.1 Preliminaries

Definition 1.1 A point on $\mathbb{R}^n$

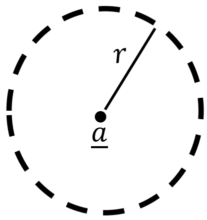
Let $D \subseteq \mathbb{R}^n$. Any ordered $n-$tuple $\underline{x} = (x_1, x_2, \cdots, x_n)$ of real numbers in D is called a 'point' in D.

For example $(1, -1)$ is a point on $\mathbb{R}^2$ and $(1, 0, 7)$ is a point on $\mathbb{R}^3$.

Definition 1.2 Open and Closed Balls

Open ball of radius r centered at $\underline{a} \in \mathbb{R}^n$ is defined as

$$B_r(\underline{a}) = \{\, \underline{x} \in \mathbb{R}^n \mid \|\underline{x} - \underline{a}\| < r\}.$$



The set of all points

$$\overline{B_r}(\underline{a}) = \{\, \underline{x} \in \mathbb{R}^n \mid \|\underline{x} - \underline{a}\| \leq r\}$$

is called the closed ball of radius r centered at $\underline{a}$.

For example, let $\underline{a} = (a, b) \in \mathbb{R}^2$. An open ball in $\mathbb{R}^2$ centered at $\underline{a}$ has the form

$$B_r(\underline{a}) = \{\underline{x} \in \mathbb{R}^2 | |\underline{x} - \underline{a}| < r\}$$
$$= \{(x, y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 < r^2\}$$

Example 1.1 Find each of the following:

(i) Open ball centered at 2 and radius 3 in $\mathbb{R}$.

(ii) Open ball centered at $(2, -1)$ and radius 1 in $\mathbb{R}^2$.

(iii) Unit closed ball centered at the origin in $\mathbb{R}^3$.

Definition 1.3 Limit Points of a set

A point $\underline{a}$ is called a limit point of the set D if and only if every open balls centered at $\underline{a}$ contain some points of D other than $\underline{a}$.

That is,

$$D \cap \left(B_r(\underline{a}) \setminus \{\underline{a}\}\right) \neq \Phi$$

for any arbitrary small $r > 0$.

Example 1.2 Find the set of all limit points of the open interval $I = (a, b)$.

Proposition 1.1 A finite set of points has no limit points.

Example 1.3 Consider the set $D = \{\underline{a}, \underline{b}, \underline{c}\}$, where $\underline{a} = (1,3), \underline{b} = (2,1), \underline{c} = (4,-1)$. Show that D has no limit points.

Proposition 1.2 Every points of $\mathbb{R}^n$ is a limit point of $\mathbb{R}^n$.

Example 1.4 Every points of $\mathbb{R}^2$ is a limit point of $\mathbb{R}^2$.

Definition 1.4 Functions of several variables

Let $D \subseteq \mathbb{R}^n$. A function $f: D \to \mathbb{R}^m$ of $n-$ variables is a rule that assigns each point $\underline{x} = (x_1, x_2, \cdots, x_n) \in D$ to a point $\underline{y} = (y_1, y_2, \cdots, y_m) \in \mathbb{R}^m$, denoted by $f(\underline{x}) = \underline{y}$. Here D is called the domain of the function f and the set of all points that f takes on $\mathbb{R}^m$ is called the range of f. That is, range of $f = \{ f(\underline{x}) \mid \underline{x} \in D \}$.

When $m = 1$, the function is called single valued function; otherwise, it is called multivalued or vector (valued) functions.

When $n = 1$, $f(x)$ is a function of single variable; otherwise, it is a function of several variables.

Example 1.5 Find the natural domain and range of the each of the following functions:
(i) $f(x,y) = x^2 + y^2$, (ii) $g(x,y) = x\ln(y^2 - x)$, (iii) $h(x,y,z) = \sqrt{9 - x^2 - y^2 - z^2}$.

EXERCICES 1

1. Find the set of all limit points of the open ball $B_r(\underline{a})$ in $\mathbb{R}^n$.
   What is the set of all limit points of the interval $[a, b)$.
2. Find the all limit points of the set $D = \left\{ \left(\frac{1}{m}, \frac{1}{n}\right) : m, n \in \mathbb{N} \right\}$.

1.2 Limit of a function

Definition 1.5 The limit point of a function

Let $\underline{a}$ be a limit point of D $(\subseteq \mathbb{R}^n)$. Then, we say that the function $f: D \to \mathbb{R}^m$ converges to a point $\underline{l} \in \mathbb{R}^m$ (or $\underline{l}$ is the limit of f at $\underline{a}$) if and only if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < \|\underline{x} - \underline{a}\| < \delta \quad \left(\text{or } \underline{x} \in B_\delta(\underline{a})\right) \implies \|f(\underline{x}) - \underline{l}\| < \varepsilon.$$

Example 1.6 Show that

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2} = 0$$

Example 1.7 Show that $\lim_{(x,y)\to(1,1)} (x^2 + xy + y^2) = 3.$

Remark: Most of the theorems and rules we had for function of single variable can be extended for functions of several variables.

1.3 Techniques of finding limits:

1.3.1 Use of Squeeze Lemma

Lemma 1.1 If $f(\underline{x}),\ g(\underline{x}),\ h(\underline{x})$ are functions such that

$$\lim_{\underline{x}\to\underline{a}} f(\underline{x}) = \lim_{\underline{x}\to\underline{a}} h(\underline{x}) = \underline{l},$$

then, $\lim_{\underline{x}\to\underline{a}} g(\underline{x})$ exists and $\lim_{\underline{x}\to\underline{a}} g(\underline{x}) = \underline{l}$.

Example 1.8 Show that

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$$

Example 1.9 Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2+2y^2}$.

Example 1.10 Evaluate $\lim_{(x,y)\to(0,0)} \sin x \sin\left(\frac{1}{x+y}\right)$.

1.3.2 Use of Polar Coordinates

Let $x = r\cos\theta$, $y = r\sin\theta\,.\sin\theta$ Then, $(x,y) \longrightarrow (0,0)$ implies that $r \longrightarrow 0^+$.

Example 1.11 Evaluate $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$.

Example 1.12 Evaluate $\lim_{(x,y)\to(1,2)} \frac{(x+y-3)^2}{\sqrt{(x-1)^2+(y-2)^2}}$.

Examples on Functions of three or more variables.

Example 1.13 Evaluate $\lim_{(x,y,z,t)\to(0,0,0,0)} \frac{(x^2+y^2)(z^2+t^2)}{(x^2+y^2+z^2+t^2)}$.

Example 1.14 Evaluate $\lim_{(x,y,z,)\to(1,0,0,)} \tan(y-xz)\cos\left(\frac{1}{(x-1)y+z^2}\right)$.

1.3.3 To Show Limit Does Not Exist

Lemma 1.2 If $\lim_{\underline{x}\to\underline{a}} f(\underline{x}) = \underline{l}$ exists, then

(i) its value $\underline{l}$ is unique, and

(ii) $\underline{l}$ is independent of the choice of any path approaching $\underline{a}$.

We use this fact to show that the limit of a function does not exist. i.e. if the limit of a function along two different Paths are not equal, then $\lim_{\underline{x}\to\underline{a}} f(\underline{x})$ does not exist.

Example 1.15 Find the limit or show that the limit does not exists:

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}.$$

Example 1.16 Find the limit or show that the limit does not exists:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}.$$

Example 1.17 Find the limit or show that the limit does not exists:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}.$$

1.4 Repeated (Iterated) limits

Definition 1.6 Let the function $f(x,y)$ is defined in the neighborhood of (a,b). Then the limit $\lim_{y\to b}\left(\lim_{x\to a} f(x,y)\right)$, if exist , is said to be repeated limits of f as $x\to a,\ y\to b$.

Remarks:

i. In general, $\lim_{y\to b}\left(\lim_{x\to a} f(x,y)\right) \neq \lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right)$.

ii. If $\lim_{(x,y)\to(a,b)} f(x,y)$ exists, then , $\lim_{y\to b}\left(\lim_{x\to a} f(x,y)\right) = \lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right)$. But the converse is not true.

iii. If $\lim_{y\to b}\left(\lim_{x\to a} f(x,y)\right) \neq \lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right)$ then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example 1.18 Show that iterated limits exist but simultaneous limit does not exist for the function

$$f(x,y) = \frac{xy}{x^2+y^2}; \quad D = \mathbb{R}\setminus\{(0,0)\}.$$

Example 1.19 Show that iterated limits exist but simultaneous limit does not exist for the function

$$f(x,y) = \begin{cases} 1, & xy \neq 0 \\ 0, & xy = 0 \end{cases}; \quad D = \mathbb{R}\setminus\{(0,0)\}.$$

Example 1.20 Verify that if iterated limits exist and equal and simultaneous limit exists, then they are equal to each other.

$$f(x,y) = \frac{3x^2y}{x^2+y^2}; \quad D = \mathbb{R}\setminus\{(0,0)\}.$$

Example 1.21 Find the iterated limits and simultaneous limit:

$$f(x,y)=\begin{cases} x\sin\left(\frac{1}{y}\right), & xy\neq 0 \\ 0, & xy=0 \end{cases}.$$

Example 1.22 Find the iterated limits and simultaneous limit:

$$f(x,y)=\begin{cases} x\sin\left(\frac{1}{y}\right)+y\sin\left(\frac{1}{x}\right), & xy\neq 0 \\ 0, & xy=0 \end{cases}.$$

1.5 Continuity of a function

Definition 1.7 The function $f\colon D\to\mathbb{R};\ D\subseteq\mathbb{R}^2$ is said to be continuous at a point $(a,b)\in D$ if and only if for each $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that $|f(x,y)-f(a,b)|<\varepsilon$ whenever $\sqrt{(x-a)^2+(y-b)^2}<\delta$.

That is, The function f is continuous at (a,b) if and only if

$$\lim_{(x,y)\to(a,b)} f(x,y)=f(a,b).$$

A point (a,b) is said to be the point discontinuity if the function f is not continuous at (a,b).

Example 1.23 Discuss the continuity of each of the following function:

(i) $f(x,y)=\frac{x-y}{1+x+y}$,

(ii) $g(x,y)=\frac{x-y}{1+x^2+y^2}$,

(iii) $h(x,y)=\frac{3x^2y}{\sin\pi x}$.

Example 1.24 Discuss the continuity of the following function

$$f(x,y)=\begin{cases} \frac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}.$$

Example 1.25 Investigate continuity of $f(x,y)$:

$$f(x,y)=\begin{cases} \frac{3xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}.$$

Example 1.26 Investigate continuity of $f(x,y)$:

$$f(x,y)=\begin{cases} \frac{\sin^2(x-y)}{|x|+|y|}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}.$$